MATH 231H SOLUTION TO SPECIAL ASSIGNMENT FOUR

Problem 1.

$$\text{length} = \int_{-10}^{10} |\mathbf{r}'(t)|dt = \int_{-10}^{10} \sqrt{(2\cos t)^2 + 5^2 + (-\sin t)^2}dt = \int_{-10}^{10} \sqrt{29}dt = 20\sqrt{29}$$

Problem 2. Point $(1, 0)$ corresponds to $t = 0$. The length of the piece of the curve corresponding to the range of parameters $[0, t]$ is

$$\begin{aligned}
s(t) &= \int_0^t \sqrt{\left(\frac{-4t}{(t^2+1)^2}\right)^2 + \left(\frac{2(t^2+1) - 4t^2}{(t^2+1)^2}\right)^2} dt \\
&= \int_0^t \frac{1}{(1+t^2)^2}\sqrt{16t^2 + (2-2t^2)^2}dt = \int_0^t \frac{1}{(1+t^2)^2}\sqrt{16t^2 + 4 - 8t^2 + 4t^4}dt \\
&= \int_0^t \frac{2}{(1+t^2)^2}\sqrt{2t^2 + 1 + t^4}dt = \int_0^t \frac{2}{(1+t^2)^2}\sqrt{(1+t^2)^2}dt \\
&= \int_0^t \frac{2dt}{1+t^2} = 2\tan^{-1}(t).
\end{aligned}$$

Therefore the time $t(s)$ when we covered distance s satisfies $2\tan^{-1} t(s) = s$, so $\boxed{t(s) = \tan \frac{s}{2}}$. The position at this time is

$$\begin{aligned}
\mathbf{r}(t(s)) &= \left(\frac{2}{\tan^2 \frac{s}{2} + 1} - 1\right)\mathbf{i} + \frac{2\tan\frac{s}{2}}{\tan^2\frac{s}{2} + 1}\mathbf{j} \\
&= (2\cos^2 \frac{s}{2} - 1)\mathbf{i} + 2\cos\frac{s}{2}\sin\frac{s}{2}\mathbf{j} \quad (\because 1 + \tan^2\theta = 1/\cos^2\theta) \\
&= (\cos s)\mathbf{i} + (\sin s)\mathbf{j} \quad (\because\ 2\cos^2\theta - 1 = \cos 2\theta\ ,\ 2\cos\theta\sin\theta = \sin 2\theta.)
\end{aligned}$$

The monstrous curve is just a circle of radius one and center at the origin!

Problem 3. Write $\mathbf{r}(t) = \langle f(t), g(t), 0\rangle$. Then

$$\begin{aligned}
\mathbf{r}' \times \mathbf{r}'' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \end{vmatrix} = (f'g'' - g'f'')\mathbf{k} \\
|\mathbf{r}'|^3 &= \left(\sqrt{(f')^2 + (g')^2 + 0^2}\right)^3 = [f'^2 + g'^2]^{3/2}
\end{aligned}$$

We get that $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = |f'g'' - f'g''|/[f'^2 + g'^3]^{3/2}$ as required.

Problem 4. This is a special case of the previous problem with $x(t) = t$, $y(t) = f(t)$.

Problem 5. Let s be the length parameter. We use primes to denote derivatives with respect to t. Recall that $s' = \frac{ds}{dt} = \frac{d}{dt}\int_a^t |\mathbf{r}'(\tau)|d\tau = |\mathbf{r}'|$. Therefore

$$\begin{aligned}
\mathbf{r}' &= |\mathbf{r}'|\frac{\mathbf{r}'}{|\mathbf{r}'|} = |\mathbf{r}'|\mathbf{T} = s'\mathbf{T}, \text{ so} \\
\mathbf{r}'' &= s''\mathbf{T} + s'\mathbf{T}' = s''\mathbf{T} + (s')^2\kappa\mathbf{N} \quad (\because \mathbf{T}' = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = s'\frac{d\mathbf{T}}{ds} = s'\kappa\mathbf{N}) \\
\mathbf{r}' \times \mathbf{r}'' &= (s')^3\kappa\mathbf{T} \times \mathbf{N} = (s')^3\kappa\mathbf{B} \\
\mathbf{r}''' &= \frac{d}{dt}\left[s''\mathbf{T} + (s')^2\kappa\mathbf{N}\right] \\
&= (s''')\mathbf{T} + (s'')s'\kappa\mathbf{N} + [(s')^2\kappa]'\mathbf{N} + (s')^2\kappa\frac{d\mathbf{N}}{dt} \\
&= (s''')\mathbf{T} + (s'')s'\kappa\mathbf{N} + [(s')^2\kappa]'\mathbf{N} + (s')^3\kappa\frac{d\mathbf{N}}{ds} \\
&= (s''')\mathbf{T} + (s'')s'\kappa\mathbf{N} + [(s')^2\kappa]'\mathbf{N} + (s')^3\kappa[-\kappa\mathbf{T} + \tau\mathbf{B}] \quad (\because \text{Serret-Frenet}) \\
&= \text{something} \cdot \mathbf{T} + \text{something} \cdot \mathbf{N} + (s')^3\kappa\tau\mathbf{B}
\end{aligned}$$

Therefore $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = (s')^6\kappa^2\tau$. Since also $|\mathbf{r}' \times \mathbf{r}''|^2 = [(s')^3\kappa]^2 = (s')^6\kappa^2$, we get that $\tau = (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''/|\mathbf{r}' \times \mathbf{r}''|^2$ as required.

Problem 6. For a general parametrization, $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{\mathbf{r}'}{v}$ so $\mathbf{r}' = v\mathbf{T}$. Differentiating again, we get

$$\begin{aligned}
\mathbf{r}'' &= (v\mathbf{T})' = v'\mathbf{T} + v\mathbf{T}' = v'\mathbf{T} + v|\mathbf{r}'|\frac{|\mathbf{T}'|}{|\mathbf{r}'|}\frac{\mathbf{T}'}{|\mathbf{T}'|} \\
&= v'\mathbf{T} + v^2\kappa\mathbf{N}, \text{ because } |\mathbf{r}'| = v, \ \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \kappa, \ \frac{\mathbf{T}'}{|\mathbf{T}'|} = \mathbf{N}.
\end{aligned}$$

Problem 7. We use the Serret-Frenet equations:

$$\begin{aligned}
\frac{d}{ds}(\mathbf{T}_1 \cdot \mathbf{T}_2) &= (\kappa\mathbf{N}_1) \cdot \mathbf{T}_2 + \mathbf{T}_1 \cdot (\kappa\mathbf{N}_2) = \kappa[\mathbf{N}_1 \cdot \mathbf{T}_2 + \mathbf{T}_1 \cdot \mathbf{N}_2] \\
\frac{d}{ds}(\mathbf{N}_1 \cdot \mathbf{N}_2) &= (-\kappa\mathbf{T}_1 + \tau\mathbf{B}_1) \cdot \mathbf{N}_2 + \mathbf{N}_1 \cdot (-\kappa\mathbf{T}_2 + \tau\mathbf{B}_2) \\
&= \kappa[-\mathbf{T}_1 \cdot \mathbf{N}_2 - \mathbf{N}_1 \cdot \mathbf{T}_2] + \tau[\mathbf{B}_1 \cdot \mathbf{N}_2 + \mathbf{N}_1 \cdot \mathbf{B}_2] \\
\frac{d}{ds}(\mathbf{B}_1 \cdot \mathbf{B}_2) &= -\tau\mathbf{N}_1 \cdot \mathbf{B}_2 + \mathbf{B}_1 \cdot (-\tau\mathbf{N}_2) \\
&= \tau[-\mathbf{N}_1 \cdot \mathbf{B}_2 - \mathbf{B}_1 \cdot \mathbf{N}_2]
\end{aligned}$$

Summing we see that $\frac{d}{ds}(\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{N}_1 \cdot \mathbf{N}_2 + \mathbf{B}_1 \cdot \mathbf{B}_2) = 0 \cdot \kappa + 0 \cdot \tau = 0$.