

MATH 231H SOLUTIONS TO SPECIAL ASSIGNMENT THREE

Problem 1. 19 is VI; 20 is II; 21 is IV; 22 is I; 23 is V; 24 is III.

Problem 2. We choose a cartesian coordinate system with the origin O at the center of the circle and the x -axis located so that at time $t = 0$ the end of the rope is at $(R, 0)$. Let $P(t) = (x(t), y(t))$ be the endpoint of the rope at time t , let $A(t)$ be the point where it touches the circle, and let $\theta(t)$ be the angle between $\overrightarrow{OA(t)}$ and the x -axis (measured counterclockwise). Then

$$\begin{aligned} \langle x(t), y(t) \rangle &= \overrightarrow{OP(t)} = \overrightarrow{OA(t)} + \overrightarrow{A(t)P(t)} \\ \overrightarrow{OA(t)} &= \langle R \cos \theta(t), R \sin \theta(t) \rangle \end{aligned}$$

To find $\overrightarrow{A(t)P(t)}$ we notice that if $\theta(t)$ is measured in *radians* then

$$\begin{aligned} \overrightarrow{A(t)P(t)} &\parallel \langle \cos(\theta(t) - \frac{\pi}{2}), \sin(\theta(t) - \frac{\pi}{2}) \rangle = \langle \sin \theta(t), -\cos \theta(t) \rangle \\ |\overrightarrow{A(t)P(t)}| &= \text{length of the arc from } (R, 0) \text{ to } A(t) = R\theta(t), \end{aligned}$$

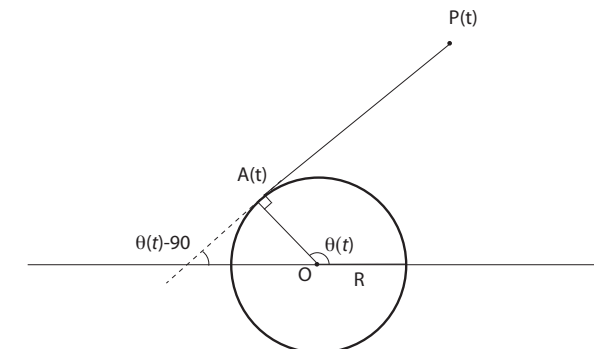
so $\overrightarrow{A(t)P(t)} = \langle R\theta(t) \sin \theta(t), -R\theta(t) \cos \theta(t) \rangle$, because this vector has the right length and direction. We conclude that

$$\langle x(t), y(t) \rangle = \langle R \cos \theta(t) + R\theta(t) \sin \theta(t), R \sin \theta(t) - R\theta(t) \cos \theta(t) \rangle.$$

It remains to find $\theta(t)$. The length of the rope that was unwrapped by time t is on the one hand $|A(t)P(t)| = R\theta(t)$ and on the other hand vt . Thus $\theta(t) = \frac{v}{R}t$. So:

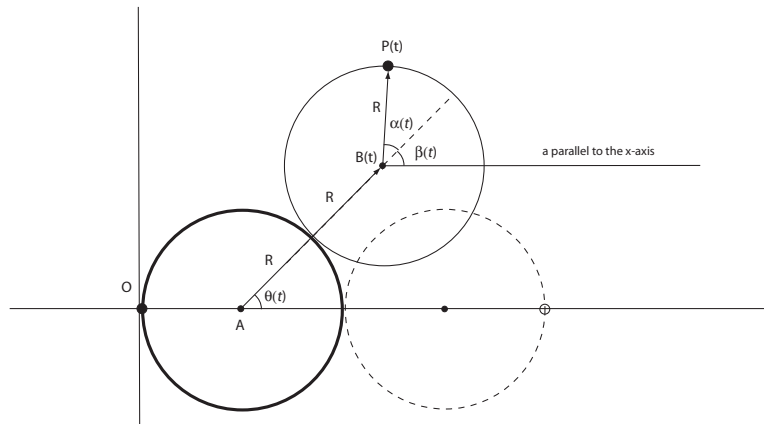
$$\langle x(t), y(t) \rangle = \langle R \cos \frac{vt}{R} + vt \sin \frac{vt}{R}, R \sin \frac{vt}{R} - vt \cos \frac{v}{R}t \rangle$$

This curve is called the *involute of the circle*. You can see an animation of this movement in <http://mathworld.wolfram.com/Involute.html> where you can also find links to further information on the relevance of this curve to 17th century clock technology – whence to navigation at sea. (Precise measurement of time was the only available method at the time for determining the longitude.)



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Problem 2



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Problem 3

Problem 3. Let $O = (0, 0)$ be the origin, $A = (R, 0)$ be the center of the fixed circle, $B(t)$ the center of the moving circle, and $P(t)$ the position of the point (all at time t). Draw a parallel to the x -axis through $B(t)$. We have

$$\begin{aligned}\overrightarrow{OP(t)} &= \overrightarrow{OA} + \overrightarrow{AB(t)} + \overrightarrow{B(t)P(t)} \\ \overrightarrow{AB(t)} &= \langle 2R \cos \theta(t), 2R \sin \theta(t) \rangle \\ \overrightarrow{B(t)P(t)} &= \langle R \cos(\alpha(t) + \beta(t)), R \sin(\alpha(t) + \beta(t)) \rangle\end{aligned}$$

where $\theta(t), \alpha(t), \beta(t)$ are as in the diagram. Now

$$\begin{aligned}\theta(t) &= \frac{vt}{R}, \text{ because } R\theta(t) = \text{distance the rolling circle moved on the fixed one} = vt \\ \alpha(t) &= \theta(t), \text{ because } P(t) \text{ moves distance } R\alpha(t) \text{ when the moving circle moves distance } R\theta(t) \\ \beta(t) &= \theta(t), \text{ because the lines are parallel.}\end{aligned}$$

We get

$$\begin{aligned}\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} &= \overrightarrow{OA} + \overrightarrow{AB(t)} + \overrightarrow{B(t)P(t)} = \begin{pmatrix} R \\ 0 \end{pmatrix} + \begin{pmatrix} 2R \cos \theta \\ 2R \sin \theta \end{pmatrix} + \begin{pmatrix} R \cos 2\theta \\ R \sin 2\theta \end{pmatrix} \\ &= \begin{pmatrix} R(1 + 2 \cos \frac{vt}{R} + \cos \frac{2vt}{R}) \\ R(2 \sin \frac{vt}{R} + \sin \frac{2vt}{R}) \end{pmatrix}.\end{aligned}$$

It is possible (but not necessary) to simplify this further using the identities $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ and $\sin 2\theta = 2 \cos \theta \sin \theta$. The result then becomes

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 2R \cos \frac{vt}{R} (1 + \cos \frac{vt}{R}) \\ 2R \sin \frac{vt}{R} (1 + \cos \frac{vt}{R}) \end{pmatrix},$$

which is nice because it means that in polar coordinates the equation of the curve are $r = 2R(1 + \cos \theta)$. This curve is called the *cardioid*, because it looks like a heart.

You can see an animation of this motion (up to reflection) at <http://mathworld.wolfram.com/Cardioid.html>, where you can also find links to information on its relation to the Mandelbrot set.

Problem 4. The curve is closed exactly when there is some some time $T_0 \neq 0$ such that $\mathbf{r}(T_0 + t) = \mathbf{r}(t)$. In coordinates, this means that

$$\begin{aligned} A_1 \sin(2\pi(T_0 + t) + \varphi_1) &= A_1 \sin(2\pi t + \varphi_1) \\ A_2 \sin(2\pi\omega(T_0 + t) + \varphi_2) &= A_2 \sin(2\pi\omega t + \varphi_2) \end{aligned}$$

The first equation implies that $2\pi T_0 = 2\pi m$ for some integer m . The second equation implies that $2\pi\omega T_0 = 2\pi n$ for some integer n . It follows that the curve is closed exactly when $\omega = n/m$ for two integers m, n which means that ω is rational.

You can find a ‘virtual oscilloscope’ for creating such curves at <http://www.coolmath.com/coolthings/lissajous/index.html>.

Problem 5 part (1). See the textbook, page 895.

Problem 5 part (2). If $\mathbf{u} = \langle u_1(t), u_2(t), u_3(t) \rangle$ and $\mathbf{v} = \langle v_1(t), v_2(t), v_3(t) \rangle$, then

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \frac{d}{dt} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \frac{d}{dt} \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \\ &= \begin{pmatrix} (u_2 v_3 - u_3 v_2)' \\ -(u_1 v_3 - u_3 v_1)' \\ (u_1 v_2 - u_2 v_1)' \end{pmatrix} = \begin{pmatrix} u_2' v_3 + u_2 v_3' - u_3' v_2 - u_3 v_2' \\ -(u_1' v_3 + u_1 v_3' - u_3' v_1 - u_3 v_1') \\ u_1' v_2 + u_1 v_2' - u_2' v_1 - u_2 v_1' \end{pmatrix} \\ &= \begin{pmatrix} u_2' v_3 - u_3' v_2 \\ -(u_1' v_3 - u_3' v_1) \\ u_1' v_2 - u_2' v_1 \end{pmatrix} + \begin{pmatrix} u_2 v_3' - u_3 v_2' \\ -(u_1 v_3' - u_3 v_1') \\ u_1 v_2' - u_2 v_1' \end{pmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1' & u_2' & u_3' \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1' & v_2' & v_3' \end{vmatrix} = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'. \end{aligned}$$

Problem 5 part (3). If $\frac{d}{dt}\langle u_1, u_2, u_3 \rangle = \langle 0, 0, 0 \rangle$, then $u_1' = u_2' = u_3' = 0$, so u_1, u_2, u_3 are constant.

Problem 6. If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ represents the location of the ant, then $\mathbf{r} \cdot \mathbf{r} = x(t)^2 + y(t)^2 + z(t)^2 = 1$. Differentiating the identity $\mathbf{r} \cdot \mathbf{r} = 1$ we get $\mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0$ (see problem 5 part (a)), whence $2\mathbf{r}' \cdot \mathbf{r} = 0$, equivalently $\mathbf{r}' \cdot \mathbf{r} = 0$. It follows that $\mathbf{v} = \mathbf{r}' \perp \mathbf{r}$.

Problem 7 part (1). Set $H(t) := K(t) + U(t)$ (total energy). We show that H is constant by checking that $H'(t) = 0$ everywhere:

$$\begin{aligned} H'(t) &= \frac{d}{dt} \left(\frac{1}{2} m |r'(t)|^2 - \frac{GMm}{|r(t)|} \right) \\ &= \frac{d}{dt} \left(\frac{1}{2} m \mathbf{r}' \cdot \mathbf{r}' - GM(\mathbf{r} \cdot \mathbf{r})^{-1/2} \right) = \frac{1}{2} m (2\mathbf{r}'' \cdot \mathbf{r}') + \frac{1}{2} GMm(\mathbf{r} \cdot \mathbf{r})^{-3/2} (2\mathbf{r}' \cdot \mathbf{r}) \\ &= \frac{1}{2} m (2\mathbf{r}'' \cdot \mathbf{r}') + \frac{1}{2} GMm(\mathbf{r} \cdot \mathbf{r})^{-3/2} (2\mathbf{r}' \cdot \mathbf{r}) \\ &= \mathbf{r}' \cdot \left[m\mathbf{r}'' + \frac{GMm}{|\mathbf{r}|^3} \mathbf{r} \right] = \mathbf{r}' \cdot \left[-\frac{GMm}{|\mathbf{r}(t)|^2} \frac{\mathbf{r}}{|\mathbf{r}|} + \frac{GMm}{|\mathbf{r}|^3} \mathbf{r} \right] \quad (\because \text{Newton's 2nd Law}) \\ &= \mathbf{r}' \cdot \mathbf{0} = 0. \end{aligned}$$

Problem 7 part (2). We show that $\mathbf{M}(t)$ is constant by showing that its derivative is everywhere equal to zero (see problem 5 part (3)):

$$\begin{aligned} \frac{d}{dt}\mathbf{M}(t) &= \frac{d}{dt}(\mathbf{r} \times \mathbf{r}') \\ &= \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' \quad (\because \text{Problem 5 part (3)}) \\ &= \mathbf{0} + \mathbf{r} \times \left(\frac{GM}{|\mathbf{r}|^3} \mathbf{r} \right) = \frac{GM}{|r|^3} \mathbf{r} \times \mathbf{r} = \mathbf{0}. \end{aligned}$$

Problem 7 part (3). The previous part says that $\mathbf{M}(t) = \mathbf{M}_0$ for some fixed vector \mathbf{M}_0 and all t . Suppose $\mathbf{M}_0 \neq \mathbf{0}$. By the definition of the angular momentum, for all t

$$\mathbf{r}(t) \perp \mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{M}(t) \equiv \mathbf{M}_0.$$

This means that the moving planet remains at the plane passing through the origin and perpendicular to \mathbf{M}_0 .

The following is not part of the exercise and is only for people who have taken a course in ODE:

If $\mathbf{M}_0 = \mathbf{0}$ then the condition $\mathbf{r} \perp \mathbf{0}$ doesn't mean anything (because it holds for all vectors), so we use a different argument: If $\mathbf{M}_0 = \mathbf{0}$, then $\mathbf{r}'(t) \times \mathbf{r}(t) = \mathbf{0}$ for all t , meaning that either $\mathbf{r}'(t) = \mathbf{0}$ or $\mathbf{r}(t) = \mathbf{0}$ or $\mathbf{r}'(t) = \lambda(t)\mathbf{r}(t)$. In the first case the particle doesn't move at all because the velocity is zero, so it's obvious that the 'motion' is planar. The same is true in the second case. In the third case we write $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and observe that $\mathbf{r}'(t) = \lambda(t)\mathbf{r}(t)$ means that

$$\begin{aligned} x'(t) &= \lambda(t)x(t) \\ y'(t) &= \lambda(t)y(t) \\ z'(t) &= \lambda(t)z(t). \end{aligned}$$

These equations can be solved: $x'(t) = \lambda(t)x(t) \Leftrightarrow \frac{x'(t)}{x(t)} = \lambda(t)$ or $x(t) = 0 \Leftrightarrow [\ln|x(t)|]' = \lambda(t)$ or $x(t) = 0 \Leftrightarrow x(t) = x(0) \exp \int_0^t \lambda(s) ds$. In the same way one solves the equations for $y(t), z(t)$ and obtains

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{\int_0^t \lambda(s) ds} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

This means that our planet is moving on a straight line passing through the origin (think of a stone thrown straight up or straight down). Certainly this motion is planar.