

MATH 231H SOLUTIONS TO SPECIAL ASSIGNMENT TWO

Problem 1.

- 21 is an ellipsoid, so it's either IV or VII. In the equation $x^2 + 4y^2 + 9z^2 = 1$ it is clear that the possible range of x -values is $[-1, 1]$ and the possible range of z -values is $[-\frac{1}{3}, \frac{1}{3}]$. We see that the ellipsoid is narrower in the z -direction. Thus the right ellipsoid is VII
- 22 is the other ellipsoid, IV.
- 23, according to the table, is a hyperboloid of one sheet, so it's II. (In the standard form mentioned in the table the signs of y, z are switched. That's why the hyperboloid is lying on its side rather than standing upright: its axis of symmetry is the axis of the variable with minus sign – the y -axis).
- 24 is a hyperboloid of two sheets with the y -axis as the axis of rotational symmetry, so it's III.
- 25 is a paraboloid, so it's VI. (III is not a paraboloid because it has two sheets. It's a hyperboloid.)
- 26 is an elliptic cone with the y -axis as the axis of rotational symmetry, so it's I.
- 27 is an elliptic cylinder, so it's VIII.
- 28 is a saddle (hyperbolic paraboloid), so it's V.

Problem 2. The angle between two planes is exactly the angle between their normals. To see this draw the intersecting planes viewed from "the front" so that their line of intersection becomes a point (see figure). If α is the angle between the planes, γ the angle between the normals, and β is as in the picture, then $\gamma = 90^\circ - \beta$, and $\beta = 90^\circ - \alpha$, so $\gamma = 90^\circ - (90^\circ - \alpha) = \alpha$.

We know that $\mathbf{n}_1 := \langle a_1, b_1, c_1 \rangle$ is normal to $a_1x + b_1y + c_1z = d_1$, and $\mathbf{n}_2 := \langle a_2, b_2, c_2 \rangle$ is normal to $a_2x + b_2y + c_2z = d_2$. So the angle α satisfies $\cos \alpha = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$

and we get $\alpha := \cos^{-1} \left(\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right)$ (measured in radians).

Remark: Note that we did not specify the *directions* of the normals to use (up/down). Different choice of direction would give 180° minus the angle we found (because $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$). Indeed, in the picture $180^\circ - \alpha$ is also an angle between the planes ...

Problem 3. Suppose $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$. Then

$$\frac{x-x_0}{a} = t \Leftrightarrow x = ta + x_0$$

$$\frac{y-y_0}{b} = t \Leftrightarrow y = tb + y_0$$

$$\frac{z-z_0}{c} = t \Leftrightarrow z = tc + z_0$$

so $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$. This means that $P(x, y, z)$ satisfies the condition in the problem if and only if $\overrightarrow{OP} = \overrightarrow{OP_0} + t\mathbf{v}$ for some real number t , where $P_0 =$

$P_0(x_0, y_0, z_0)$ and $\mathbf{v} = \langle a, b, c \rangle$. This is exactly saying that P is on the line passing through P_0 in the direction \mathbf{v} .

Thus the equation is the equation of the line passing through (x_0, y_0, z_0) , in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$.

Problem 4 (part 1). X lies on the plane generated by A, B, C exactly when \overrightarrow{AX} is perpendicular to the normal to the plane generated by A, B, C . Here is a formula for a normal to this plane: $\overrightarrow{AB} \times \overrightarrow{AC}$. We get the vectorial equation

$$\overrightarrow{AX} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0.$$

Since $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA} = \mathbf{x} - \mathbf{a}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$, and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{c} - \mathbf{a}$, we get the equation $(\mathbf{x} - \mathbf{a}) \cdot ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) = 0$.

Problem 4 (part 2). Calculating we see that

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \langle -1, 0, 2 \rangle \times \langle -2, -1, 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \langle 2, -3, 1 \rangle,$$

so the equation becomes $\langle x-1, y, z+1 \rangle \cdot \langle 2, -3, 1 \rangle = 0$, or $2(x-1) - 3y + (z+1) = 0$, or $2x - 3y + z = 1$.

Problem 5. Define a cartesian coordinate system whose z -axis is the axis of rotational symmetry of the cylinder, and the origin at the center of the cylinder, so that the 'rings' are at heights $+h, -h$. (The orientation of the x, y axes is not important.) In our coordinate systems

$$\begin{aligned} \text{Top Ring} &= \{(r \cos \alpha, r \sin \alpha, h) : 0 \leq \alpha < 2\pi\} \\ \text{Ring Bottom} &= \{(r \cos \alpha, r \sin \alpha, -h) : 0 \leq \alpha < 2\pi\} \end{aligned}$$

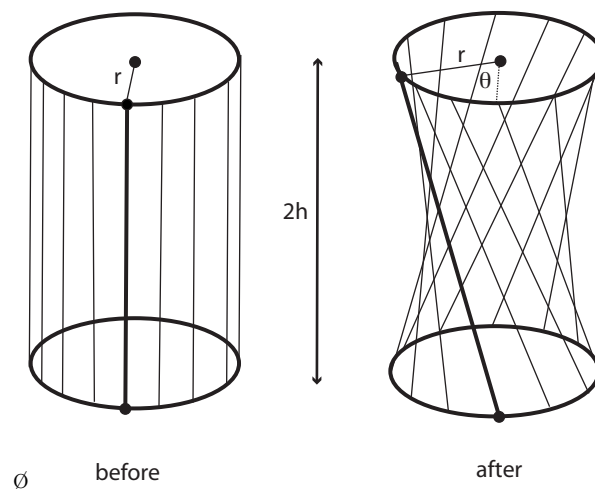
The idea is to find a quadric equation relating the x, y, z -coordinates of a general point $P(x, y, z)$ on the surface.

If $P(x, y, z)$ is on our surface, then it sits on one of the skewed lines in the picture. Before rotating the top ring, this line was a vertical, connecting two points of the form $(r \cos \alpha, r \sin \alpha, h)$, $(r \cos \alpha, r \sin \alpha, -h)$. After rotation of the top ring this line becomes the line connecting $(r \cos(\alpha + \theta), r \sin(\alpha + \theta), h)$ to $(r \cos \alpha, r \sin \alpha, -h)$. Any point on this line has the form

$$(r \cos \alpha, r \sin \alpha, -h) + t(r \cos(\alpha + \theta) - r \cos \alpha, r \sin(\alpha + \theta) - r \sin \alpha, 2h), \quad (t \in \mathbb{R}).$$

In particular, if (x, y, z) is on our surface, then

$$\begin{aligned} x &= (1-t)r \cos \alpha + tr \cos(\alpha + \theta) \\ y &= (1-t)r \sin \alpha + tr \sin(\alpha + \theta) \\ z &= (2t-1)h \end{aligned}$$



From now on we just play with the equations seeking to annihilate t, α .

$$\begin{aligned}
 x^2 + y^2 &= (1-t)^2 r^2 + 2r^2 t(1-t)[\cos \alpha \cos(\theta + \alpha) + \sin \alpha \sin(\theta + \alpha)] + t^2 r^2 \\
 &= (1-t)^2 r^2 + 2r^2 t(1-t) \cos(\theta + \alpha - \alpha) + t^2 r^2 \\
 &= r^2[(1-t)^2 + 2(1-t)t \cos \theta + t^2] = r^2[(1-t)^2 + 2(1-t)t + t^2] + 2r^2(1-t)t(\cos \theta - 1) \\
 &= r^2 + 2r^2(1-t)t(\cos \theta - 1)
 \end{aligned}$$

$$t = \frac{1}{2} \left(\frac{z}{h} + 1 \right) = \frac{z+h}{2h}$$

$$(1-t)t = \frac{2h - (z+h)}{2h} \cdot \frac{z+h}{2h} = \frac{h^2 - z^2}{4h^2}$$

Substituting this in the equation for $x^2 + y^2$ we get

$$x^2 + y^2 = r^2 + 2r^2(1 - \cos \theta) \frac{z^2 - h^2}{4h^2}, \text{ or}$$

$$x^2 + y^2 - \frac{r^2}{2h^2}(1 - \cos \theta)z^2 = r^2 - \frac{r^2}{2}(1 - \cos \theta)$$

and this is the equation of a hyperboloid of one sheet.

If $\theta = 180^\circ$, then we get $x^2 + y^2 - \frac{r^2}{h^2}z^2 = 0$, or $z = \pm \frac{h}{r} \sqrt{x^2 + y^2}$, the equation of a cone. Can you visualize the hyperboloid degenerating into a cone?