MATH 231H SOLUTIONS TO SPECIAL ASSIGNMENT ONE

Problem 1. Let M be the midpoint. Then (refer to the diagram on the assignment sheet):

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ \overrightarrow{OM} &= \overrightarrow{OB} + \overrightarrow{BM}\\ \overrightarrow{AM} &= -\overrightarrow{BM} \ (\because |\overrightarrow{AM}| = |\overrightarrow{BM}| \text{ and they have opposite directions}).\end{aligned}$$

Summing the first two equations gives $2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} \Rightarrow \boxed{\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})}$.

Problem 2 part (1). In the diagram on the assignment sheet:

$$\begin{aligned}\overrightarrow{AP} &= t\overrightarrow{AB} \quad (\because \tfrac{|AP|}{|AB|} = t)\\ \overrightarrow{AQ} &= t\overrightarrow{AC} \quad (\because \tfrac{|AQ|}{|AC|} = t)\\ \overrightarrow{PQ} &= \overrightarrow{AQ} - \overrightarrow{AP} \quad (\because \overrightarrow{AP} + \overrightarrow{PQ} = \overrightarrow{AQ})\\ \overrightarrow{BC} &= \overrightarrow{AC} - \overrightarrow{AB} \quad (\because \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC})\end{aligned}$$

Thus $\overrightarrow{PQ} = t\overrightarrow{AC} - t\overrightarrow{AB} = t(\overrightarrow{AC} - \overrightarrow{AB}) = t\overrightarrow{BC}$. which exactly means that $\frac{|PQ|}{|BC|} = t$ (same size) and that $PQ \| BC$ (same direction).

Problem 2 part (2). The assumptions in part (2) can be expressed in vector notation by the equation

$$\overrightarrow{PQ} = t\overrightarrow{BC}.$$

Also, since $\overrightarrow{AP}, \overrightarrow{AB}$ have the same direction, and $\overrightarrow{AQ}$, $\overrightarrow{AC}$ have the same direction we have by the definition of the scalar multiplication that

$$\begin{aligned}\overrightarrow{AP} &= \alpha\overrightarrow{AB} \text{ where } \alpha := \frac{|AP|}{|AB|}\\ \overrightarrow{AQ} &= \beta\overrightarrow{AC} \text{ where } \beta := \frac{|AQ|}{|AC|}\end{aligned}$$

The idea is to show that $\alpha = \beta = t$. As in the solution of part (1), $\overrightarrow{PQ} = \overrightarrow{AQ} - \overrightarrow{AP}$, $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AP}$, so

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{AC} - \overrightarrow{AP} = \frac{1}{\beta}\overrightarrow{AQ} - \frac{1}{\alpha}\overrightarrow{AB}\\ \overrightarrow{BC} &= \frac{1}{t}\overrightarrow{PQ} = \frac{1}{t}\overrightarrow{AQ} - \frac{1}{t}\overrightarrow{AP}\end{aligned}$$

which means that two right hand sides of the above equations are equal. After some rearrangement, this gives $\left(\frac{1}{\beta} - \frac{1}{t}\right)\overrightarrow{AQ} = \left(\frac{1}{t} - \frac{1}{\alpha}\right)\overrightarrow{AP}$. The vectors $\overrightarrow{AQ}, \overrightarrow{AP}$ have different directions, so they cannot be proportional. This means that the terms in the brackets vanish , whence $\frac{1}{\beta} = \frac{1}{t}, \frac{1}{t} = \frac{1}{\alpha} \Rightarrow \alpha = \beta = t$ as we wanted.

Problem 3. Let $ABCD$ be a parallelogram (with the vertices listed counterclockwise), and set $\mathbf{a} = \overrightarrow{AB}, \mathbf{b} = \overrightarrow{AD}$. The other sides of the parallelogram are

$$\overrightarrow{BC} = \overrightarrow{AD} = \mathbf{a}\ ,\ \overrightarrow{DC} = \overrightarrow{AB} = \mathbf{b}.$$

Its diagonals are

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \mathbf{a} + \mathbf{b}\ ,\ \overrightarrow{DB} = \overrightarrow{DC} - \overrightarrow{BC} = \mathbf{b} - \mathbf{a}$$

What we are asked to prove, in vector notation, is $2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 = |\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2$. We start from the LHS:

$$\begin{aligned} LHS &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \big(|\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + |\mathbf{b}|^2\big) + \big(|\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + |\mathbf{b}|^2\big) \\ &= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 = RHS. \end{aligned}$$

Problem 4 part (1). It is enough to prove the squared inequality $|\mathbf{a}+\mathbf{b}|^2 \leq (|\mathbf{a}| + |\mathbf{b}|)^2$:

$$\begin{aligned} RHS &= |\mathbf{a}|^2 + 2|\mathbf{a}|\cdot|\mathbf{b}| + |\mathbf{b}|^2 \\ &\geq |\mathbf{a}|^2 + 2|\mathbf{a}|\cdot|\mathbf{b}|\cos\measuredangle(\mathbf{a},\mathbf{b}) + |\mathbf{b}|^2 \\ &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\ &= |\mathbf{a}+\mathbf{b}|^2 = LHS \end{aligned}$$

In a triangle $\triangle ABC$, the sides have lengths $|\overrightarrow{AB}|, |\overrightarrow{BC}|, |\overrightarrow{AC}|$ and we see that $|\overrightarrow{AB}| + |\overrightarrow{BC}| \geq |\overrightarrow{AB} + \overrightarrow{BC}| = |\overrightarrow{AC}|$. How can we see that the inequality is strict (i.e. $>$, not just $\geq$)? Studying the chain of inequalities above we see that the only way for the RHS to equal the LHS is when $2|\overrightarrow{AB}|\cdot|\overrightarrow{BC}| = 2|\overrightarrow{AB}|\cdot|\overrightarrow{BC}|\cos\measuredangle(\overrightarrow{AB},\overrightarrow{BC})$ which means that $\measuredangle ABC = 0°$ or $180°$. This cannot happen in a triangle, so the $\geq$ can be replaced by $>$.

Problem 4 part (2). Part (1) implies

$$\begin{aligned} |\mathbf{a}-\mathbf{b}| + |\mathbf{b}| \geq |\mathbf{a}| &\quad\Rightarrow\quad |\mathbf{a}-\mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}| \\ |\mathbf{b}-\mathbf{a}| + |\mathbf{a}| \geq |\mathbf{b}| &\quad\Rightarrow\quad |\mathbf{a}-\mathbf{b}| = |\mathbf{b}-\mathbf{a}| \geq |\mathbf{b}| - |\mathbf{a}| \end{aligned}$$

This means $-|\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a}-\mathbf{b}|$, equivalently $\big||\mathbf{a}| - |\mathbf{b}|\big| \leq |\mathbf{a}-\mathbf{b}|$.

Problem 5. Working with 2D vectors we see that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \mathbf{a}\cdot\mathbf{x}$ where $\mathbf{x} = \langle b_2, -b_1\rangle$. What is this vector $\mathbf{x}$?

- Length: $|\mathbf{x}| = \sqrt{b_2^2 + (-b_1)^2} = |\mathbf{b}|$;
- Direction: $\mathbf{x}\cdot\mathbf{b} = b_2b_1 - b_1b_2 = 0$ so $\mathbf{x} \perp \mathbf{b}$. Thus $\measuredangle(\mathbf{a},\mathbf{x}) = \measuredangle(\mathbf{a},\mathbf{b}) \pm 90°$ (can you draw a picture where the minus sign is the correct one?)

Consequently $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \mathbf{a}\cdot\mathbf{x} = |\mathbf{a}|\cdot|\mathbf{b}|\cos(\measuredangle(\mathbf{a},\mathbf{b}) \pm 90°) = \pm|\mathbf{a}|\cdot|\mathbf{b}|\sin\measuredangle(\mathbf{a},\mathbf{b})$. The latter expression is the area of a parallelogram with sides $|\mathbf{a}|, |\mathbf{b}|$ (cut the parallelogram along the height from the endpoint of $\mathbf{b}$ to side $\mathbf{a}$, rearrange the pieces to form a rectangle, and calculate its area).

Problem 6.

(1) Let $\theta := \measuredangle(\mathbf{a}, \mathbf{b})$.

$$\begin{aligned} RHS &= |\mathbf{a}|^2|\mathbf{b}|^2 - (|\mathbf{a}||\mathbf{b}|\cos\theta)^2 \quad (\because \text{definition of dot product}) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta) = \big(|\mathbf{a}||\mathbf{b}|\sin\theta\big)^2 \\ &= |\mathbf{a}\times\mathbf{b}|^2 = LHS \ (\because \text{definition of cross product}) \end{aligned}$$

(2) Write $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Those who do not know determinants should then just calculate using coordinates $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and see that they get the same thing at the end. Those that know determinants can bypass the calculation by noticing that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

(3) I don't know a clever way of doing this, except via direct calculation using coordinates. To keep the equations of reasonable length I will prove the equation in the special cases $\mathbf{a} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ and then note that these imply the general case because both sides of the identity depend on $\mathbf{a}$ in a linear way.

$\mathbf{i}\times(\mathbf{b}\times\mathbf{c}) =$

$$\begin{aligned} &= \mathbf{i}\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= (\mathbf{i}\times\mathbf{i})\cdot\text{something} - (\mathbf{i}\times\mathbf{j})(b_1c_3 - b_3c_1) + (\mathbf{i}\times\mathbf{k})(b_1c_2 - b_2c_1) \\ &= -\mathbf{k}(b_1c_3 - b_3c_1) - \mathbf{j}(b_1c_2 - b_2c_1) \quad (\because\ \mathbf{i}\times\mathbf{i} = \mathbf{0}, \mathbf{i}\times\mathbf{j} = \mathbf{k}, \mathbf{i}\times\mathbf{k} = -\mathbf{j}) \\ &= (b_2c_1 - b_1c_2)\mathbf{j} + (b_3c_1 - b_1c_3)\mathbf{k} \end{aligned}$$

$(\mathbf{i}\cdot\mathbf{c})\mathbf{b} - (\mathbf{i}\cdot\mathbf{b})\mathbf{c} =$

$$\begin{aligned} &= c_1\mathbf{b} - b_1\mathbf{c} = (c_1b_1 - b_1c_1)\mathbf{i} + (c_1b_2 - b_1c_2)\mathbf{j} + (c_1b_3 - b_1c_3)\mathbf{k} \\ &= (b_2c_1 - b_1c_2)\mathbf{j} + (b_3c_1 - b_1c_3)\mathbf{k} \end{aligned}$$

so $\mathbf{i}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{i}\cdot\mathbf{c})\mathbf{b} - (\mathbf{i}\cdot\mathbf{b})\mathbf{c}$. Similar calculations establish the identity for $\mathbf{a} = \mathbf{j}, \mathbf{k}$. The general case now follows from the linearity of the cross and dot products:

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) =$

$$\begin{aligned} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(\mathbf{b}\times\mathbf{c}) \\ &= [a_1\mathbf{i}\times(\mathbf{b}\times\mathbf{c})] + [a_2\mathbf{j}\times(\mathbf{b}\times\mathbf{c})] + [a_3\mathbf{k}\times(\mathbf{b}\times\mathbf{c})] \\ &= [(a_1\mathbf{i}\cdot\mathbf{c})\mathbf{b} - (a_1\mathbf{i}\cdot\mathbf{b})\mathbf{c}] + [(a_2\mathbf{j}\cdot\mathbf{c})\mathbf{b} - (a_2\mathbf{j}\cdot\mathbf{b})\mathbf{c}] + [(a_3\mathbf{k}\cdot\mathbf{c})\mathbf{b} - (a_3\mathbf{k}\cdot\mathbf{b})\mathbf{c}] \\ &= ((a_1\mathbf{i} + a_2\mathbf{j} + a_3 + k)\cdot\mathbf{c})\mathbf{b} - ((a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot\mathbf{b})\mathbf{c} \\ &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \end{aligned}$$

Problem 7.

(1) Proof of *Jacobi's identity*: We use Problem 6 (c). Each summand in $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ is a linear combination of two of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We calculate the total coefficient of each of these vectors:
   - $\boxed{\text{Coefficient of } \mathbf{a}\text{:}}$ $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$ contributes $-\mathbf{b} \cdot \mathbf{c}$ and $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ contributes $\mathbf{c} \cdot \mathbf{b}$. Total: $-\mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{b} = 0$.
   - $\boxed{\text{Coefficient of } \mathbf{b}\text{:}}$ $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ contributes $\mathbf{a} \cdot \mathbf{c}$, $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ contributes $-\mathbf{c} \cdot \mathbf{a}$. Total: $\mathbf{a} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a} = 0$.
   - $\boxed{\text{Coefficient of } \mathbf{c}\text{:}}$ $-\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} = 0$.

   Thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0\mathbf{a} + 0\mathbf{b} + 0\mathbf{c} = \mathbf{0}$.

(2) $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$, but $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{k} = \mathbf{0}$.

(3) Note that

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\ &= -\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \quad (\because \text{ Jacobi's identity}) \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &\Leftrightarrow \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0} \Leftrightarrow |\mathbf{b} \times (\mathbf{c} \times \mathbf{a})| = 0 \\ &\Leftrightarrow |\mathbf{b}| \cdot |\mathbf{c} \times \mathbf{a}| \sin \measuredangle(\mathbf{b}, \mathbf{c} \times \mathbf{a}) = 0 \\ &\Leftrightarrow |\mathbf{b}||\mathbf{c}||\mathbf{a}| \sin \measuredangle(\mathbf{b}, \mathbf{a}) \sin \measuredangle(\mathbf{b}, \mathbf{c} \times \mathbf{a}) = 0 \end{aligned}$$

In other words, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ if and only at least one of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is equal to zero, or $\mathbf{b} || \mathbf{a}$, or $\mathbf{c} || \mathbf{a}$, or $\mathbf{b} \perp \mathbf{a}, \mathbf{c}$.

Problem 8. In a triangle $\triangle ABC$, the height to AC is $|AB| \sin \measuredangle CAB$ so the area is $\frac{1}{2}|AC| \cdot |AB| \sin \measuredangle CAB = |\overrightarrow{AC}||\overrightarrow{AB}| \sin \measuredangle(\overrightarrow{AB}, \overrightarrow{AC}) = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

Problem 9. Consider a regular polygon with $2n$ sides (n even) with some arbitrary point P inside it. Let O denote the center of the polygon (i.e. the center of the circumscribing circle). Label the vertices of P from V_1 to V_{2n} counterclockwise. We define the following vectors (see figure below):

- $\mathbf{a}_i = \overrightarrow{PV_i}$ $(i = 1, \ldots, 2n)$.
- $\mathbf{b}_i = \overrightarrow{OV_i}$ $(i = 1, \ldots, 2n)$.
- $\mathbf{c}_i = \overrightarrow{V_i V_{i+1}}$ $(i = 1, \ldots, 2n)$. Note:

$$\begin{aligned} \mathbf{c}_i &= \mathbf{b}_{i+1} - \mathbf{b}_i \\ \mathbf{c}_i &= -\mathbf{c}_j \text{ whenever } |i - j| = n \text{ (see figure).} \end{aligned}$$

- $\mathbf{r} = \overrightarrow{OP}$ (so $\mathbf{a}_i = \mathbf{r} + \mathbf{b}_i$).

We assume that our polygon is in the xy–plane. Let A_i denote the area of $\triangle PV_iV_{i+1}$, and let S denote the total area of the polygon. By problem 8, $\frac{1}{2}\mathbf{a}_i \times \mathbf{a}_{i+1} = A_i\mathbf{k}$, so it is enough to prove that

$$\mathbf{a}_2 \times \mathbf{a}_3 + \mathbf{a}_4 \times \mathbf{a}_5 + \cdots + \mathbf{a}_{2n-2} \times \mathbf{a}_{2n-1} + \mathbf{a}_{2n} \times \mathbf{a}_1 = S\mathbf{k}$$

(this would mean that the areas of the 'even' triangles is half the total area).

Note that if $P = O$, then this is indeed the case, because in that case all triangles have the same area. In other words, we always have the equation

$$\mathbf{b}_2 \times \mathbf{b}_3 + \mathbf{b}_4 \times \mathbf{b}_5 + \cdots + \mathbf{b}_{2n-2} \times \mathbf{b}_{2n-1} + \mathbf{b}_{2n} \times \mathbf{b}_1 = S\mathbf{k} \qquad (*)$$

Recalling that $\mathbf{a}_i = \mathbf{r} + \mathbf{b}_i$ we see that

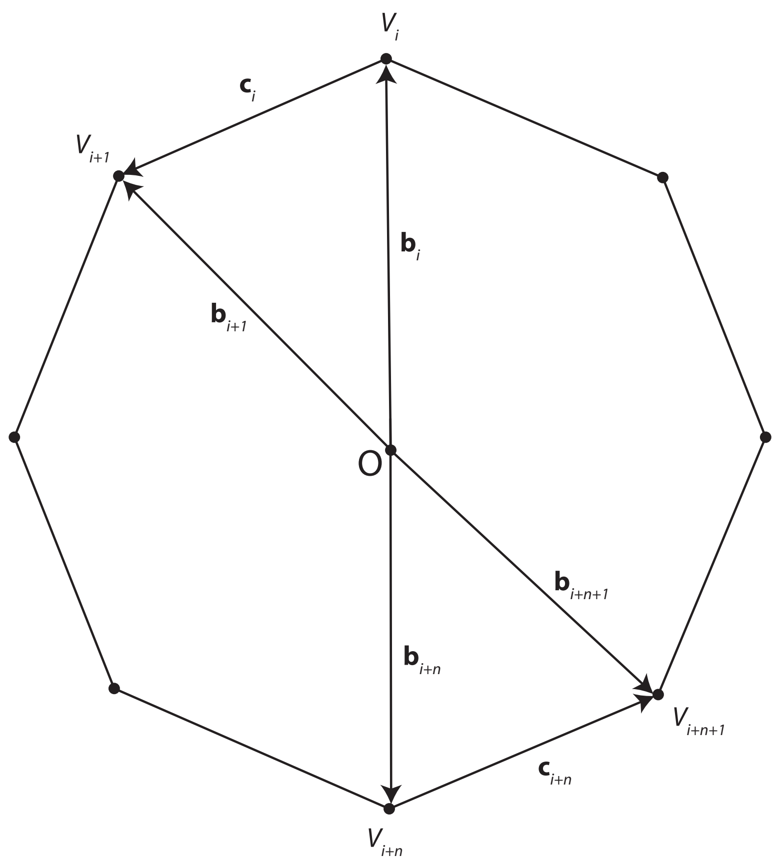
$$\begin{aligned}
&\mathbf{a}_2 \times \mathbf{a}_3 + \mathbf{a}_4 \times \mathbf{a}_5 + \cdots + \mathbf{a}_{2n-2} \times \mathbf{a}_{2n-1} + \mathbf{a}_{2n} \times \mathbf{a}_1 = \\
&= \quad (\mathbf{r} + \mathbf{b}_2) \times (\mathbf{r} + \mathbf{b}_3) + \cdots + (\mathbf{r} + \mathbf{b}_{2n}) \times (\mathbf{r} + \mathbf{b}_1) \\
&= \quad n\mathbf{r} \times \mathbf{r} + \mathbf{r} \times (\mathbf{b}_1 + \mathbf{b}_3 + \cdots + \mathbf{b}_{2n-1}) + (\mathbf{b}_2 + \mathbf{b}_4 + \cdots + \mathbf{b}_{2n}) \times \mathbf{r} \\
&\qquad + (\mathbf{b}_2 \times \mathbf{b}_3 + \mathbf{b}_4 \times \mathbf{b}_5 + \cdots + \mathbf{b}_{2n-2} \times \mathbf{b}_{2n-1} + \mathbf{b}_{2n} \times \mathbf{b}_1) \\
&= \quad \mathbf{0} + [(\mathbf{b}_2 - \mathbf{b}_1) + (\mathbf{b}_4 - \mathbf{b}_3) + \cdots + (\mathbf{b}_{2n} - \mathbf{b}_{2n-1})] \times \mathbf{r} + S\mathbf{k}, \quad \text{by } (*) \\
&= \quad [\mathbf{c}_1 + \mathbf{c}_3 + \cdots + \mathbf{c}_{2n-1}] \times \mathbf{r} + S\mathbf{k}
\end{aligned}$$

If n is even, then for every $\mathbf{c}_i$ in the brackets one can find a $\mathbf{c}_j$ in the brackets such that $|i - j| = n$, whence by the above discussion $\mathbf{c}_i + \mathbf{c}_j = \mathbf{0}$ (Look at the picture: can you identify for each side of the polygon the side which 'cancells' it?). We see that the sum in the brackets vanishes and so

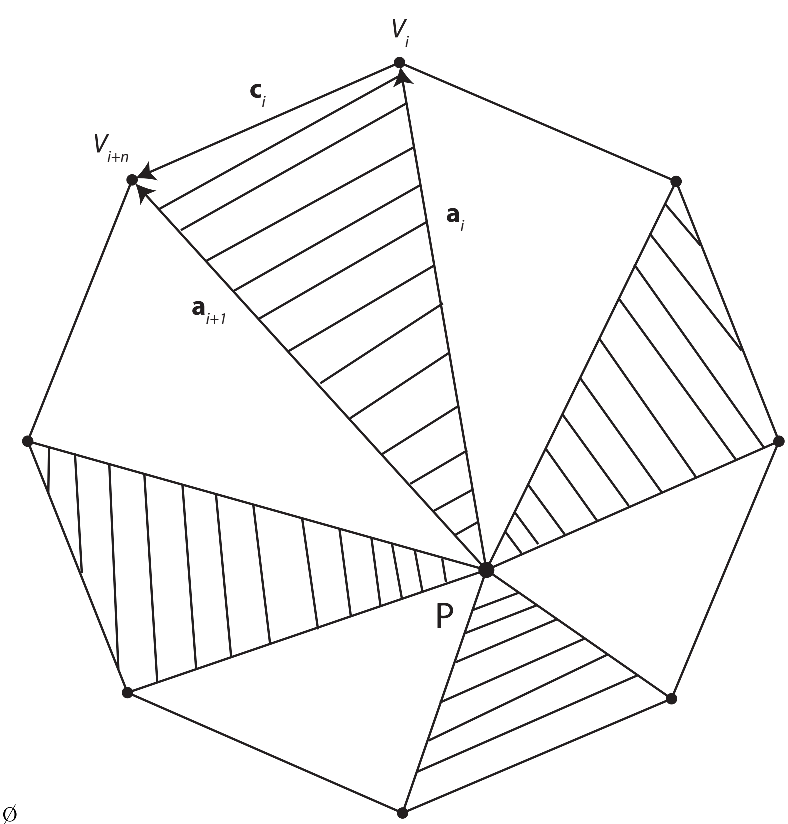
$$\mathbf{a}_2 \times \mathbf{a}_3 + \mathbf{a}_4 \times \mathbf{a}_5 + \cdots + \mathbf{a}_{2n-2} \times \mathbf{a}_{2n-1} + \mathbf{a}_{2n} \times \mathbf{a}_1 = S\mathbf{k}$$

as required.

Remark: In fact it can be shown that the condition that n is even is not necessary, and that the sum in the brackets always vanishes. Can you do this? (Hint: complex numbers help here)



V_i
$\mathbf{c}_i$
V_{i+1}
$\mathbf{b}_i$
$\mathbf{b}_{i+1}$
O
$\mathbf{b}_{i+n+1}$
$\mathbf{b}_{i+n}$
V_{i+n+1}
$\mathbf{c}_{i+n}$
V_{i+n}



V_i
$\mathbf{c}_i$
V_{i+n}
$\mathbf{a}_i$
$\mathbf{a}_{i+1}$
P

∅