MATH 231H SPECIAL ASSIGNMENT FOUR

Submit on April 3. Please prove all your assertions.

Problem 1. Do problem 1 in section 14.3 .

Problem 2. Do problem 12 in section 14.3.

Problem 3. Show that the curvature of plane curve given in parametric form $x = f(t), y = g(t)$ is given by

$$\kappa = \frac{|f'(t)g''(t) - f''(t)g'(t)|}{[f'(t)^2 + g'(t)^2]^{3/2}}.$$

Problem 4. Show that the curvature of the graph $y = f(x)$ at the point (x, y) is $\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}}$.

Problem 5. Prove that $\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$. (See exercise 51 in section 14.3 for a hint)

Problem 6. The motion of a particle is described by a function $\mathbf{r}(t)$. Show that its acceleration is given by $\mathbf{r}''(t) = a_T\mathbf{T} + a_N\mathbf{N}$ where

$$\begin{aligned} a_T &= v', \text{ where } v = \text{speed} = |r'| \\ a_N &= \kappa v^2, \text{ where } \kappa = \text{curvature at point } \mathbf{r}. \end{aligned}$$

Deduce: the component of $\mathbf{r}''$ in the direction of $\mathbf{T}$ is responsible to changes in speed; the component of $\mathbf{r}''$ in the direction of $\mathbf{N}$ is responsible to changes in direction; the torsion of the motion is a 3rd derivative effect.

Problem 7. Let $\mathbf{r}_1(s)$, $\mathbf{r}_2(s)$ be two smooth curves parametrized by length, such that $\mathbf{r}_1', \mathbf{r}_2' \neq \mathbf{0}$. Let $\mathbf{T}_i, \mathbf{N}_i, \mathbf{B}_i$ $(i = 1, 2)$ be the associated tangen, normal, and binormal vectors. Suppose $\mathbf{r}_1, \mathbf{r}_2$ have the same curvature and torsion functions. prove the claim stated in class that

$$\frac{d}{ds}\big(\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{N}_1 \cdot \mathbf{N}_2 + \mathbf{B}_1 \cdot \mathbf{B}_2\big) = 0$$