MATH 231H SPECIAL ASSIGNMENT THREE

Submit on March 18th. Please prove all your assertions.

Problem 1. Do problems 19–24 in section 14.1, on page 891 of the textbook.

Problem 2. A rope is wrapped around a cylinder of radius R. Describe the motion of the end of the rope if it is unfolded at constant speed v meters/seconds by pulling it tightly, see figure.

Problem 3. A circle of radius R is placed on the x-axis with its center at $(R, 0)$. A second circle of radius R is placed next to it with it center at $(3R, 0)$. Point $(4R, 0)$ on the second circle is colored black. Describe its motion if the second circle rolls at constant speed v on the first circle, see figure.

Problem 4. Let $A_1, A_2, \varphi_1, \varphi_2$ be non-zero constants. The curve $\mathbf{r}(t) = \langle A_1 \sin(2\pi t + \varphi_1), A_2 \sin(2\pi\omega t + \varphi_2)\rangle$ describes the motion of a particle whose x and y coordinates oscillate independently (such curves can be simulated on an oscilloscope). Prove:

(1) If ω is rational[1], then this curve is closed;
(2) If ω is irrational, then this curve does not close on itself.

Problem 5. Suppose $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$ where $u_i(t), v_i(t)$ are differentiable. Verify the following:

(1) $\frac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$
(2) $\frac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$
(3) If $\frac{d}{dt}\mathbf{u}(t) = \mathbf{0}$ for all t, then $\mathbf{u}$ is constant. (You may accept as given this property for scalar valued functions.)

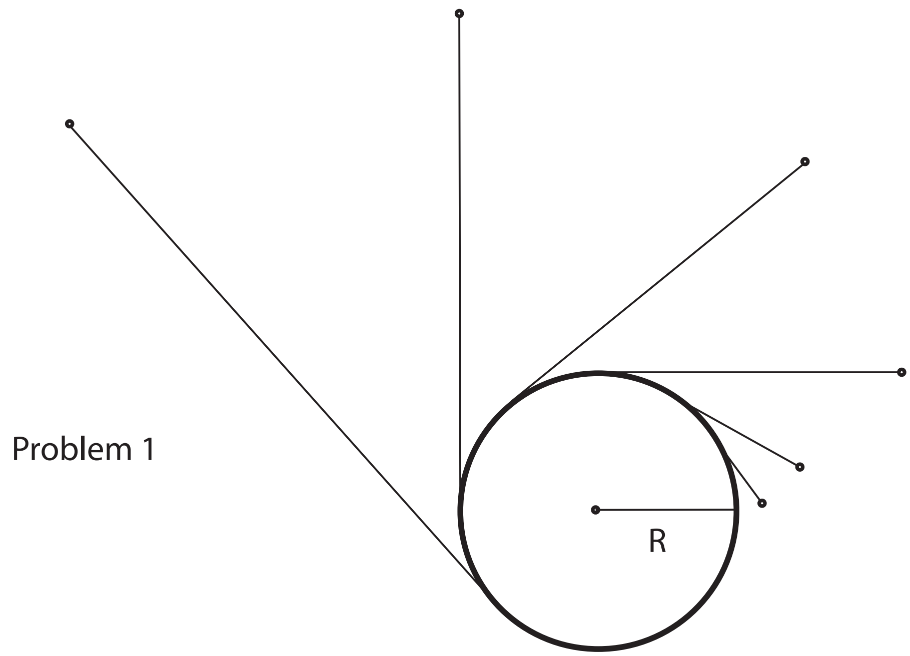
Problem 6. An ant is walking on the face of the sphere $x^2 + y^2 + z^2 = 1$. Suppose its location and velocity at time t are $\mathbf{r}(t)$, $\mathbf{v}(t)$. Prove that $\mathbf{v}(t) \perp \mathbf{r}(t)$ at all times.

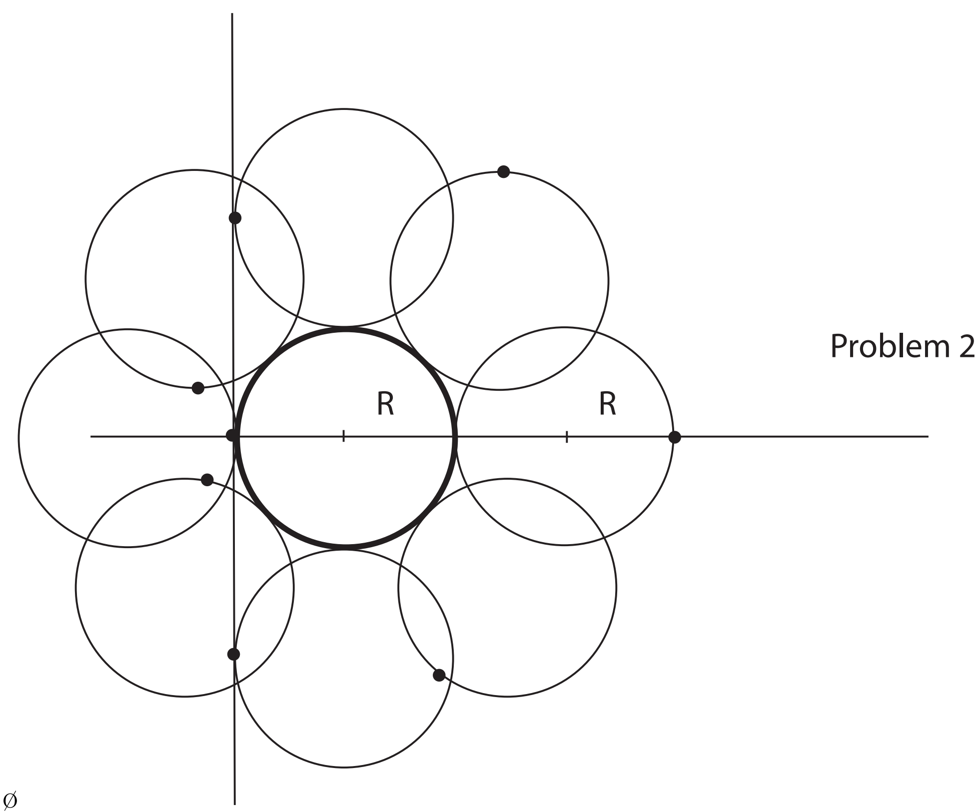
Problem 7. A planet with mass m is moving under the effect of the gravitation of another planet of mass M which stays fixed at the origin. Let $\mathbf{r}(t)$ be the position of the planet. Newton's second law of motion says

$$m\mathbf{r}''(t) = -\frac{GMm}{|\mathbf{r}(t)|^2}\frac{r(t)}{|r(t)|} \quad \text{(RHS=graviational force)}$$

(1) The *Kinetic energy* of the moving planet is $K(t) := \frac{1}{2}m|r'(t)|^2$. The *(Gravitational) potential energy* os the moving planet is $U(t) = -\frac{GMm}{|\mathbf{r}(t)|}$. Prove: $K(t) + U(t)$ is constant. ('Conservation of energy')
(2) The *angular momentum* of the moving planet is $\mathbf{M}(t) := \mathbf{r}(t) \times \mathbf{r}'(\mathbf{t})$. Prove: $\mathbf{M}(t)$ is constant. ('Conservation of angular momentum')
(3) Use part (2) to prove that the orbit of planet m is completely contained in some plane. ('Planar motion in gravitational field'). You may assume that the angular momentum is not zero. (If you know ODE, don't assume this)

[1] A rational number is a number of the form m/n with m, n integers. An irrational number is a number which cannot be put in this form, e.g. $\sqrt{2}$.





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