MATH 231H SPECIAL ASSIGNMENT ONE

Submit on 2/14. Please prove all your assertions. Your grade is based on problems 1–8. If you do problem 9, I will give you a bonus towards the final grade.

Problem 1. Suppose O, A, B are points not on the same line and set $\mathbf{a} := \overrightarrow{OA}$, $\mathbf{b} := \overrightarrow{OB}$. Let M be the point bisecting AB. Express $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (Prove your formula)

Problem 2. Let $\triangle ABC$ be a triangle, P be a point on AB, and Q a point on AC. Prove using vectors the following fundamental geometric facts:

(1) If $\frac{|AP|}{|AB|} = \frac{|AQ|}{|AC|} = t$, then $PQ \| BC$ and $|PQ| = t|BC|$. (Hint: Express $\overrightarrow{PQ}$ in terms of $\overrightarrow{AB}$, $\overrightarrow{AC}$ and t.)

*(2) If $PQ \| BC$ and $|PQ| = t|BC|$, then $\frac{|AP|}{|AB|} = \frac{|AQ|}{|AC|} = t$. (Hint: Denote $\alpha := \frac{|AP|}{|AB|}, \beta := \frac{|AQ|}{|AC|}$ and find a vector equation involving α, β, t).

Problem 3. Prove using scalar products that the sums of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of its diagonals.

Problem 4. Let $\mathbf{a}, \mathbf{b}$ be vectors. Prove the following inequalities:

(1) *Triangle inequality:* $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. Deduce: In a triangle, the sum of two sides is larger than the other side.

(2) *Reverse triangle inequality:* $|\mathbf{a} - \mathbf{b}| \geq \big||\mathbf{a}| - |\mathbf{b}|\big|$.

Hint: Deduce (2) from (1).

Problem 5. In $\mathbb{R}^2$, the area of the parallelogram spanned by vectors $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$ is (up to a sign) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. (Hint: Write the determinant in the form $\mathbf{a} \cdot \mathbf{x}$ with $\mathbf{x} = \langle x_1, x_2 \rangle$ and identify $\cos \measuredangle(\mathbf{a}, \mathbf{x})$.)

Problem 6. Prove the following identities:

(1) $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a} \cdot \mathbf{b}|^2$

(2) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(3) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

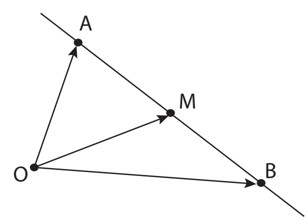
Problem 7.

(1) Prove *Jacobi's identity*: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

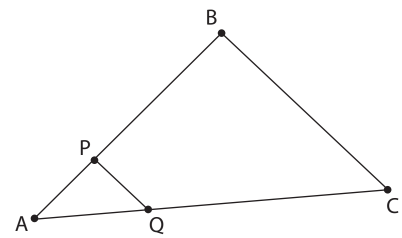
(2) Give an example of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ showing that the vector product is not associative.

(3) Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-zero vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. What can you say on $\mathbf{a}, \mathbf{b}, \mathbf{c}$?

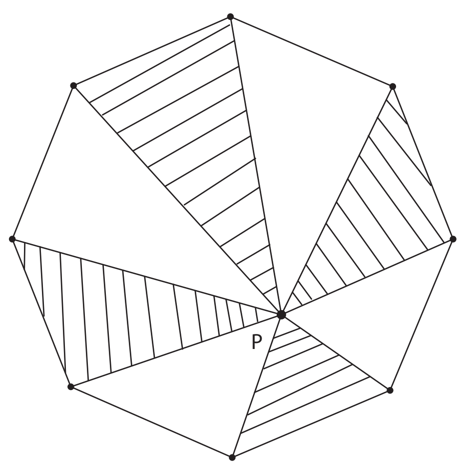
Problem 8. The area of a triangle $\triangle ABC$ is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.



Problem 1



Problem 2



∅ Problem 9

***Problem 9.** Consider a regular polygon with $2n$ sides. Let P be an arbitrary point inside the polygon. Connect P to the vertices of the polygon thus dividing it into $2n$ triangles. Color every other triangle black (see figure). Show, using cross products, that the total area of the black triangles is equal to the total area of the white triangles. You may assume that n is even.