MATH 017 SECTIONS 006 AND 007
ALTERNATIVE PROOF OF THE EQUIVALENCE OF A STATEMENT AND ITS CONTRAPOSITIVE.

We want to show that the two statements $p \to q$ and $(\neg q) \to (\neg p)$ are logically equivalent. We thus need to check that they have the same truth values in all possible cases.

We first consider the cases in which $p \to q$ is true, and then the cases in which $p \to q$ is false. This obviously exhausts all the possible cases.

1. Assume that $p \to q$ is true. $(\bigotimes)$

   Note that $(\neg q) \to (\neg p)$ is a well defined statement.

   Since $(\neg q) \to (\neg p)$ is a statement, it is either true or false.

   We will show that it is impossible for it to be false, and it will then follow that $(\neg q) \to (\neg p)$ is true.

   Suppose $(\neg q) \to (\neg p)$ was false. Since it is an implication, the only way in which it could be false would be if the premise was true but the consequent was false. The premise is $(\neg q)$ and the consequent is $(\neg p)$. If the premise $(\neg q)$ is true, then q *is false*. If the consequent $(\neg p)$ is false, then p *is true*. Now consider what this means for the truth of $p \to q$. We now have that if $(\neg q) \to (\neg p)$ is false, then p, the premise of $p \to q$, is true. If $(\neg q) \to (\neg p)$ is false, then q, the consequent of $p \to q$, is false. But now $p \to q$ has a true premise and a false consequent, and is hence a false implication. But this contradicts our assumption that $p \to q$ was true in the first place (refer to $(\bigotimes)$).

   Hence $(\neg q) \to (\neg p)$ cannot be false. Hence $(\neg q) \to (\neg p)$ must be true when $p \to q$ is.

2. Next suppose that $p \to q$ is false. This happens exactly when p is true and q is false. This is the same as to say that $(\neg p)$ is false and $(\neg q)$ is true. But now the statement $(\neg q) \to (\neg p)$ has a true premise and a false consequent, and is therefore false.

   Hence $(\neg q) \to (\neg p)$ is false when $p \to q$ is false.

It follows now that the statements $p \to q$ and $(\neg q) \to (\neg p)$ are indeed logically equivalent.

The proof is complete (QED).

Please send questions, corrections or comments to *sanyal@math.psu.edu* .