Math 597e Primes, Spring 2008, Solutions 7

Let $f(\alpha)$ be as in the lectures, let $Q = N^{1/2}(\log N)^{-B}$ and $P = (\log N)^D$ where B and D will be fixed later. Further let $K(\alpha) = |\sum_{h=1}^{H} e(\alpha h)|^2 = \sum_{|h|\le H}(H - |h|)e(\alpha h)$ be the Fejér kernel, where we suppose that $H \ll \log N$. We now define the major arc $\mathfrak{M}(q,a) = [a/q - P/N, a/q + P/N]$ and take $\mathfrak{M}$ to be the union of the $\mathfrak{M}(q,a)$ with $1 \le a \le q \le Q$ and $(a,q) = 1$. Also let $\mathfrak{U} = (P/N, 1+P/N]$ and $R(N,h) = \sum_{p_1,p_1}(\log p_1)(\log p_2)$ where the sum is over primes p_1, p_2 with $p_j \le N$ and $p_2 - p_1 = h$.

Note, there are a number of corrections.

1. Prove that $\int_{\mathfrak{U}} |f(\alpha)|^2 K(\alpha) d\alpha = \sum_{h=-H}^{H}(H - |h|)R(N,h)$. By orthogonality the integral is $\sum_{p_1\le N}\sum_{p_2\le N}(\log p_1)(\log p_2)\sum'_{|h|\le H}(H-|h|)$ where the sums are over those p_1, p_2, h such that $p_2 - p_1 = h$.

2. Prove that $R(N,0) = N\log N + O(N)$. LHS $= \sum_{p\le N}(\log p)^2 = (\log N)\vartheta(N) - \int_2^N \vartheta(u)u^{-1}du$.

3. Prove that $\int_{\mathfrak{U}} |f(\alpha)|K(\alpha)d\alpha \ll H^2(N\log N)^{1/2}$. The integral is $\ll H^2\int_{\mathfrak{U}}|f(\alpha)|\alpha \ll H^2\left(\int_{\mathfrak{U}}|f(\alpha)|^2\alpha\right)^{1/2} = H^2 R(N,0)^{1/2}$.

4. Show that if $\alpha \in \mathfrak{M}(q,a)$, then $K(\alpha) = K(a/q) + O(PH^2N^{-1})$. Deduce that $HN\log N + 2\sum_{h=1}^{H}(H - h)R(N,h) \ge I_1 + O(HN)$ where $I_1 = \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q} K(a/q)\int_{\mathfrak{M}(q,a)}|f(\alpha)|^2 d\alpha$. We have $K(\alpha) - K(a/q) = \sum_{|h|\le H}(H-|h|)\int_{a/q}^{\alpha} 2\pi i e(h\gamma)d\gamma \ll H^2|\alpha - a/q| \ll H^2PN^{-1}$, so the integral is $\ge \int_{frakM}|f(\alpha)|^2K(\alpha)d\alpha = I_1 + O\left(|f(\alpha)|^2H^2PN^{-1}\right)$. Thus by Q1, 2, $HN\log N + O(HN) + 2\sum_{h=1}^{H}(H-h)R(N,h) \ge I_1 + O(H^2P\log N)$.

5. Let $f_q(\alpha) = \sum_{\substack{p\le N\\ p\nmid q}}(\log p)e(\alpha p)$. Prove that $f(\alpha) - f_q(\alpha) \ll \log q$ and that $I_1 = I_2 + O\big(N^{1/2}(\log N)^4\big)$ where I_2 is as I_1 but with f replaced by f_q. Let $E_q(\alpha) = f(\alpha) - f_q(\alpha)$. Then $E_q(\alpha) \ll \sum_{p|q}\log p = \log q$. Thus $|f_q(\alpha)|^2 = |f(\alpha) - E_q(\alpha)|^2 = |f(\alpha)|^2 - 2\Re\big(f(\alpha)\overline{E_q(\alpha)} + |E_q(\alpha)|^2$. Hence $|f(\alpha)|^2 - |f_q(\alpha)|^2 \ll |f(\alpha)|\log q + (\log q)^2$ and so $I_1 - I_2 \ll H^2(\log Q)\int_{\mathfrak{U}}|f(\alpha)|d\alpha + H^2(\log Q)^2\int_{\mathfrak{U}}d\alpha \ll (\log N)^3(N\log N)^{1/2} + (\log N)^4$.

6. Define $V(\alpha)$ by $V(\alpha) = \mu(q)\phi(q)^{-1}\sum_{n=1}^{N} e\big((\alpha - a/q)\big)$ when $\alpha \in \mathfrak{M}(q,a)$ and by 0 when $\alpha \notin \mathfrak{M}$. Let $E(\alpha) = f_q(\alpha) - V(\alpha)$. Prove that $|f_q(\alpha)|^2 \ge |V(\alpha)|^2 + 2\Re\big(V(\alpha)\overline{E(\alpha)}\big)$ and that $I_2 \ge I_3 + 2\Re I_4$ where I_3, I_4 are as I_1 but with $|f|^2$ replaced by $|V|^2$, $V\overline{E}$ resp. $|f_q(\alpha)|^2 = |V(\alpha)|^2 + 2\Re\big(V(\alpha)\overline{E(\alpha)}\big) + |E(\alpha)|^2$, etc.

7. Prove that $\int_{\mathfrak{M}(q,a)}|V(\alpha)|^2 = \mu(q)^2\phi(q)^{-2}\big(N + O(N/P)\big)$ and that with $\mathfrak{S}(h,Q)$ as in the lectures $I_3 = I_5 + O\big(H^2(\log Q)NP^{-1}\big)$ where $I_5 = N\sum_{|h|\le H}(H - |h|)\mathfrak{S}(h,Q)$. We have $\int_{\mathfrak{M}(q,a)}\left|\sum_{n=1}^{N} e(\alpha - a/q)\right|^2 d\alpha = \int_{-1/2}^{1/2}\left|\sum_{n=1}^{N} e(\beta)\right|^2 d\beta + O\left(\int_{P/N}^{1/2}\beta^{-2}d\beta\right)$. Hence $I_3 - I_5 \ll \sum_{q\le Q}\sum_{a=1,(a,q)=1}^{q} H^2\mu(q)^2\phi(q)^{-2}NP^{-1} \ll H^2(\log Q)NP^{-1}$.

In summary, provided that $D \ge 3$, $HN\log N + 2\sum_{h=1}^{H}(H - h)R(N,h) \ge I_5 + 2\Re I_4 + O(HN)$.

8. Let $c(n,q,r) = \log n - 1/\phi(q)$ when n is prime and $n \equiv r \pmod q$, and $c(n,q,r) = -1/\phi(q)$ otherwise, and let $\Delta(\beta,q,r) = \sum_{n\le N} c(n,q,r)e(\beta n)$. Prove that if $\alpha \in \mathfrak{M}(q,a)$, $\alpha = a/q + \beta$, then $E(\alpha) = \sum_{\substack{r=1\\(r,q)=1}}^{q}\Delta(\beta,q,r)e(ar/q)$. Deduce that if $S(r) = \sum_{\substack{a=1\\(a,q)=1}}^{q} K(a/q)e(ar/q)$ and $I(r) = \int_{-P/N}^{P/N}|\sum_{n=1}^{N} e(n\beta)||\Delta(\beta,q,r)|d\beta$, then $|I_4| \le \sum_{q\le Q}\mu(q)^2\phi(q)^{-1}\sum_{\substack{r=1\\(r,q)=1}}^{q}|S(r)|I(r)$. We have $E(\alpha) = \sum_{r=1,(r,q)=1}^{q} e(ar/q)\left(\sum_{p\le N, p\equiv r \pmod q}(\log p)e(\beta p) - \sum_{n\le N} e(\beta n)/\phi(q)\right)$, etc. Hence

$$I_4 = 2\Re\left(\sum_{q\le Q}\mu(q)/\phi(q)\sum_{r=1,(r,q)=1}^{q}\sum_{a=1,(a,q)=1}^{q} K(a/q)e(-ar/q)\int_{-P/N}^{P/N}\sum_{n=1}^{N} e(n\beta)\Delta(-\beta,q,r)d\beta\right).$$