MATH 597E PRIMES, SPRING 2008, SOLUTIONS 5

Let A be fixed and positive, N be large, $Q = (\log N)^B$ with $B = B(A)$, and $\mathfrak{M}(q,a)$, $\mathfrak{M}$, $\mathfrak{m}$, f, $\mathfrak{S}(h,Q)$, $\mathfrak{S}(h)$ be as in the lectures. Assume any result from the lectures that may be useful.

1. Prove that $\int_{\mathfrak{M}} |f(\alpha)|^2 e(h\alpha)d\alpha = (N-|h|)\mathfrak{S}(h,Q) + O\left(N(\log N)^{-A}\right)$ when $|h| \le N$.

When $\alpha \in \mathfrak{M}(q,a)$ we have $f(\alpha) = \frac{\mu(q)}{\phi(q)} \sum_{n=1}^N e(\beta n) + O\big(N\exp(-c\sqrt{\log N}\big)$. Thus $|f(\alpha)|^2 = \mu(q)^2\phi^{-2}(q)\left|\sum_{n=1}^N e(\beta n)\right|^2 + O\big(N^2\exp(-c\sqrt{\log N}\big)$. Integrating over $[a/q - Q/N, a/q + Q/N]$ and summing over $1 \le a \le q \le Q$ with $(a,q) = 1$ gives $\int_{\mathfrak{M}} |f(\alpha)|^2 e(h\alpha)d\alpha = \mathfrak{S}(h;Q)I_1(h) + O\big(N\exp(-c'\sqrt{\log N})\big)$ with $I_1(h) = \int_{-Q/N}^{Q/N} \left|\sum_{m=1}^N e(\beta m)\right|^2 e(h\beta)d\beta$. The sum in the integrand here is $\ll \|\beta\|^{-1}$ and so one can replace the interval of integration by $[-1/2, 1/2]$ with an error $\ll N/Q$, and we have $|\mathfrak{S}(h;Q) \ll \log Q$. Moreover the new integral is the number of choices of m_1, m_2 with $1 \le m_j \le N$ and $m_2 - m_1 = h$, and this is $N - |h|$.

2. Prove that $\sum_{h\in\mathbb{Z}} \left|\int_{\mathfrak{m}} |f(\alpha)|^2 e(\alpha h)d\alpha\right|^2 \ll N^3(\log N)^{-A}$.

The integral here is the Fourier coefficient of the function which is $|f(\alpha)|^2$ on $\mathfrak{m}$ and 0 elsewhere. Thus by Bessel's inequality the expression on the left is $\le \int_{\mathfrak{m}} |f(\alpha)|^4 d\alpha$ and now one can proceed exactly as in the lectures.

3. (Lavrik 1961.) Prove that $\sum_{h=1}^N |R(N,h) - (N-h)\mathfrak{S}(h)|^2 \ll N^3(\log N)^{-A}$ where $R(N,h) = \sum_{\substack{p_1,p_2 \le N \\ p_2 - p_1 = h}} \log p_1 \log p_2$.

We have $R(N,h) = \int_0^1 |f(\alpha)|^2 e(h\alpha)d\alpha = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right)|f(\alpha)|^2 e(h\alpha)d\alpha$. Thus, when $|h| \le N$ we have $|R(N,h) - \mathfrak{S}(h)(N-|h|)|^2 \ll S_1(h) + S_2(h) + S_3(h)$ where $S_1(h) = N^2|\mathfrak{S}(h) - \mathfrak{S}(h;Q)|^2$, $S_2(h) = \left|\int_{\mathfrak{M}} |f(\alpha)|^2 e(h\alpha)d\alpha - (N-|h|)\mathfrak{S}(h;Q)\right|^2$ and $S_3(h) = \left|\int_{\mathfrak{m}} |f(\alpha)|^2 e(h\alpha)d\alpha\right|^2$. As in the lectures we have $\mathfrak{S}(h) - \mathfrak{S}(h;Q) \ll d(h)Q^{-1/2}$ ($h \ne 0$) and so $\sum_{h=1}^N S_1(h) \ll N^3(\log N)^3 Q^{-1}$. The conclusion then follows from questions 1 and 2.