Math 597e Primes, Spring 2008, Solutions 3

1. Recall that $\sum_{x=1}^q \chi(x)e(xn/q) = \overline{\chi}(n)\tau(\chi)$ holds for all χ modulo q when $(n,q)=1$. (a) Show that if $a_n = 0$ whenever $(n,q)>1$ then $\sum_\chi |\tau(\chi)|^2|S(\chi)|^2 = \varphi(q)\sum_{\substack{a=1\\(a,q)=1}}^q |T(a/q)|^2$. (b) Suppose that $a_n = 0$ whenever n has a prime factor $\le Q$. Show that $\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi \pmod q}|\tau(\chi)|^2|S(\chi)|^2 \le \Lambda(N,Q)\sum_{n=M+1}^{M+N}|a_n|^2$. (c) Suppose that $a_n = 0$ whenever $(n,q)>1$, and that the character $\chi \pmod q$ is induced by the primitive character $\chi^* \pmod d$. Show that $S(\chi)=S(\chi^*)$. Recall also that $\tau(\chi)=\tau(\chi^*)\mu(q/d)$ if $(q/d,d)=1$ and that $\tau(\chi)=0$ otherwise. (d) Let $\sum^*_\chi$ denote summation over the primitive characters modulo q. Show that if the a_n are as in (b) then $\sum_{q\le Q}\frac{q}{\varphi(q)}\Big(\sum^*_\chi |S(\chi)|^2\Big)\Big(\sum_{\substack{k\le Q/q\\(k,q)=1}}\frac{\mu(k)^2}{\varphi(k)}\Big) \le \Lambda(N,Q)\sum_{n=M+1}^{M+N}|a_n|^2$. (e) Show that if the a_n are as in (b) then $\sum_{q\le Q}\big(\log Q/q\big)\sum^*_\chi |S(\chi)|^2 \le \Lambda(N,Q)\sum_{n=M+1}^{M+N}|a_n|^2$.
(f) Let $\mathcal{N}$ be the set of those integers $n\in[M+1,M+N]$ such that $(n,q)=1$ for all $q\le Q$. Put $Z=\operatorname{card}\mathcal{N}$. Show that
$Z^2\log Q + \sum_{1<q\le Q}\big(\log Q/q\big)\sum^*_\chi\Big|\sum_{n\in\mathcal{N}}\chi(n)\Big|^2 \le \Lambda(N,Q)Z.$

(a) We have $|\tau(\chi)| = |\tau(\overline{\chi})|$ and so

$$|\tau(\chi)S(\chi)| = \left|\tau(\overline{\chi})\sum_n a_n\chi(n)\right| = \left|\sum_n a_n\sum_{x=1}^q\overline{\chi}(x)e(xn/q)\right| = \left|\sum_{x=1}^q\overline{\chi}(x)T(x/q)\right|.$$

At once by orthogonality, $\sum_{\chi \pmod q}|\tau(\chi)|^2|S(\chi)|^2 = \varphi(q)\sum_{\substack{x=1\\(x,q)=1}}^q |T(x/q)|^2$.

(b) Divide both sides of (a) by $\varphi(q)$, sum over $q\le Q$ and apply the large sieve.

(c) When $(n,q)=1$, so that $(n,d)=1$ also, we have $\chi(n)=\chi^*(n)$.

(d) Correction, $\tau(\chi)=\tau(\chi^*)\mu(q/d)\chi^*(q/d)$. Now in the LHS of (b), $|S(\chi) = S(\chi^*)$ when χ is induced by the primitive character χ^*. Thus the sum over χ in (b) is 0 unless $(d,q/d)=1$, in which case it is

$$\sum_{d|q,(d,q/d)=1}\ \sum^*_{\chi^* \pmod d}\mu(q/d)^2|\tau(\chi^*)|^2||S(\chi^*)|^2.$$

Replacing q on the LHS of our sum by dk gives the double sum

$$\sum_{dk\le Q,(d,k)=1}\ \sum^*_{\chi^* \pmod d}\frac{|\tau(\chi^*)|^2}{\varphi(dk)}|S(\chi^*)|^2.$$

and the conclusion follows on noting that $|\tau(\chi^*)|^2 = d$ and applying (b).

(e) From estimates in lectures we have $\sum_{k\le X,(k,q)=1}\mu(k)^2/\varphi(q) \ge \varphi(q)q^{-1}\log X$.

(f) In (e) the term $q=1$, so that $\chi\equiv 1$, gives $Z^2\log Q$.