

**Math 597e Primes, Spring 2008, Problems 7**

**Due Tuesday 18th March**

This and the next homework prove that  $\liminf \frac{p_{n+1}-p_n}{\log p_n} \leq \frac{1}{2}$  (Bombieri & Davenport, 1965). Let  $f(\alpha)$  be as in the lectures, let  $Q = N^{1/2}(\log N)^{-B}$  and  $P = (\log N)^D$  where  $B$  and  $D$  will be fixed later. Further let  $K(\alpha) = |\sum_{h=1}^H e(\alpha h)|^2 = \sum_{|h| \leq H} (H - |h|)e(\alpha h)$  be the Fejér kernel, where we suppose that  $H \ll \log N$ . We now define the major arc  $\mathfrak{M}(q, a) = [a/q - P/N, a/q + P/N]$  and take  $\mathfrak{M}$  to be the union of the  $\mathfrak{M}(q, a)$  with  $1 \leq a \leq q \leq Q$  and  $(a, q) = 1$ . Also let  $\mathfrak{U} = (P/N, 1 + P/N]$  and  $R(N, h) = \sum_{\substack{p_1, p_2 \\ p_j \leq N \\ p_2 - p_1 = h}} (\log p_1)(\log p_2)$  where the sum is over primes  $p_1, p_2$  with  $p_j \leq N$  and  $p_2 - p_1 = h$ . You may assume anything proved in class or in earlier homeworks.

1. Prove that  $\int_{\mathfrak{U}} |f(\alpha)|^2 K(\alpha) d\alpha = \sum_{h=-H}^H (H - |h|) R(N, h)$ .
2. Prove that  $R(N, 0) = N \log N + O(N)$ .
3. Prove that  $\int_{\mathfrak{U}} |f(\alpha)| K(\alpha) d\alpha \ll H^2 (N \log N)^{1/2}$ .
4. Show that if  $\alpha \in \mathfrak{M}(q, a)$ , then  $K(\alpha) = K(a/q) + O(PH^3 N^{-1})$ . Deduce that  $HN \log N + 2 \sum_{h=1}^H (H - h) R(N, h) \geq I_1 + O(HN)$  where

$$I_1 = \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q K(a/q) \int_{\mathfrak{M}(q,a)} |f(\alpha)|^2 d\alpha.$$

5. Let  $f_q(\alpha) = \sum_{\substack{p \leq N \\ p \nmid q}} (\log p) e(\alpha p)$ . Prove that  $f(\alpha) - f_q(\alpha) \ll (\log N) \log 2q$  and that  $I_1 = I_2 + O(N^{1/2}(\log N)^3)$  where  $I_2$  is as  $I_1$  but with  $f$  replaced by  $f_q$ .
6. Define  $V(\alpha)$  by  $V(\alpha) = \mu(q)\phi(q)^{-1} \sum_{n=1}^N e((\alpha - a/q)n)$  when  $\alpha \in \mathfrak{M}(q, a)$  and by 0 when  $\alpha \notin \mathfrak{M}$ . Let  $E(\alpha) = f_q(\alpha) - V(\alpha)$ . Prove that  $|f_q(\alpha)|^2 \geq |V(\alpha)|^2 + 2\Re(V(\alpha)\overline{E(\alpha)})$ . Deduce that  $I_2 \geq I_3 + 2\Re I_4$  where  $I_3, I_4$  are as  $I_1$  but with  $|f|^2$  replaced by  $|V|^2$  and  $V\overline{E}$  respectively.
7. Prove that  $\int_{\mathfrak{M}(q,a)} |V(\alpha)|^2 = \mu(q)^2 \phi(q)^{-2} (N + O(N/P))$  and deduce that with  $\mathfrak{S}(h, Q)$  as in the lectures  $I_3 = I_5 + O(H^2(\log Q)NP^{-1})$  where  $I_5 = N \sum_{|h| \leq H} (H - |h|) \mathfrak{S}(h, Q)$ .

In summary, provided that  $D \geq 2$ ,

$$HN \log N + 2 \sum_{h=1}^H (H - h) R(N, h) \geq I_5 + 2\Re I_4 + O(HN).$$

8. Let  $c(n, q, r) = \log n - 1/\phi(q)$  when  $n$  is prime and  $n \equiv r \pmod{q}$ , and  $c(n, q, r) = -1/\phi(q)$  otherwise, and let  $\Delta(\beta, q, r) = \sum_{n \leq N} c(n, q, r)$ . Prove that if  $\alpha \in \mathfrak{M}(q, a)$ ,  $\alpha = a/q + \beta$ , then  $E(\alpha) = \sum_{\substack{r=1 \\ (r,q)=1}}^q \Delta(\beta, q, r)$ . Deduce that if  $S(r) = \sum_{\substack{a=1 \\ (a,q)=1}}^q K(a/q) e(ar/q)$  and  $I(r) = \int_{-P/N}^{P/N} |\sum_{n=1}^N e(n\beta)| |\Delta(\beta, q, r)| d\beta$ , then

$$|I_4| \leq \sum_{q \leq Q} \mu(q)^2 \phi(q)^{-1} \sum_{\substack{r=1 \\ (r,q)=1}}^q |S(r)| I(r).$$