# MATH 597E PRIMES, SPRING 2008, PROBLEMS 6

**Due Tuesday 4th March**

Let $S(x)=\sum_{n\le x}\frac{1}{\phi(n)}$, $U(x)=\sum_{p\le x} d(p+1)$ and suppose that $x\ge 2$.

1. (i) Prove that $\frac{1}{\phi(n)}=\frac{1}{n}\sum_{m|n}\frac{\mu(m)^2}{\phi(m)}$.

(ii) Prove that $S(x)=\sum_{m\le x}\frac{\mu(m)^2}{m\phi(m)}\sum_{r\le x/m}\frac{1}{r}$.

(iii) Prove that $S(x)=C_1\log x+C_2+E$ where $E\ll\frac{1}{x}\sum_{m\le x}\frac{\mu(m)^2}{\phi(m)}+\sum_{m>x}\frac{\mu(m)^2\log m}{m\phi(m)}$ and $C_1=\sum_{m=1}^{\infty}\frac{\mu(m)^2}{m\phi(m)}$ and $C_2=C_1\gamma-\sum_{m=1}^{\infty}\frac{(\log m)\mu(m)^2}{m\phi(m)}$ with $\gamma$ Euler's constant.

(iv) Prove that $S(x)\ll\log x$ and deduce that $E\ll(\log x)/x$.

(v) Prove that $C_1=\frac{\zeta(2)\zeta(3)}{\zeta(6)}$ and $C_2=C_1\left(\gamma-\sum_p\frac{\log p}{p^2-p+1}\right)$.

(vi) Deduce that $\sum_{n\le x}\frac{n}{\phi(n)}\ll x$ (the above argument can be adapted to show that this sum is $C_1x+O(\log x)$).

2. (i) Prove that $U(x)=\sum_{q\le\sqrt{x}}\pi(x;q,-1)+\sum_{r\le\sqrt{x}}\left(\pi(x;r,-1)-\pi(r\sqrt{x};r,-1)\right)+O(\sqrt{x})$.

(ii) Prove that $\sum_{r\le\sqrt{x}}\pi(r\sqrt{x};r,-1)\ll\sum_{r\le\sqrt{x}}\frac{r\sqrt{x}}{\phi(r)\log x}\ll\frac{x}{\log x}$ (the Brun–Titchmarsh theorem and 2(vi) are relevant).

(iii) Let $Q=x^{1/2}(\log x)^{-B}$ where $B$ is a suitable constant to be fixed later. Show that $\sum_{Q<q\le\sqrt{x}}\pi(x;q,-1)\ll_B\frac{x}{\log x}\log\log x$ (Brun–Titchmarsh and 2(iv) are useful).

(iv) Prove that $\sum_{q\le Q}\pi(x;q,-1)=S(Q)\mathrm{li}(x)+O_B(x/\log x)$ (here Bombieri-Vinogradov is useful).

(v) Prove that $U(x)=C_1x+O\left(\frac{x}{\log x}\log\log x\right)$.