MATH 597E PRIMES, SPRING 2008, PROBLEMS 3

Due Tuesday 12th February

Let $S(\chi)=\sum_{n=M+1}^{M+N}a_n\chi(n)$, $T(\alpha)=\sum_{n=M+1}^{M+N}a_ne(n\alpha)$.

1. Recall that

$$\sum_{x=1}^{q}\chi(x)e(xn/q)=\overline{\chi}(n)\tau(\chi)$$

holds for all χ modulo q when $(n,q)=1$.
(a) Show that if $a_n=0$ whenever $(n,q)>1$ then

$$\sum_{\chi}|\tau(\chi)|^2|S(\chi)|^2=\varphi(q)\sum_{\substack{a=1\\(a,q)=1}}^{q}|T(a/q)|^2.$$

(b) Suppose that $a_n=0$ whenever n has a prime factor $\le Q$. Show that

$$\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi\ (\mathrm{mod}\ q)}|\tau(\chi)|^2|S(\chi)|^2\le\Lambda(N,Q)\sum_{n=M+1}^{M+N}|a_n|^2.$$

(c) Suppose that $a_n=0$ whenever $(n,q)>1$, and that the character χ (mod q) is induced by the primitive character χ^* (mod d). Show that $S(\chi)=S(\chi^*)$. Recall also that $\tau(\chi)=\tau(\chi^*)\mu(q/d)\chi^*(q/d)$ if $(q/d,d)=1$ and that $\tau(\chi)=0$ otherwise.
(d) Let $\sum^*_{\chi}$ denote summation over the primitive characters modulo q. Show that if the a_n are as in (b) then

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\Big(\sum_{\chi}{}^{*}|S(\chi)|^2\Big)\Big(\sum_{\substack{k\le Q/q\\(k,q)=1}}\frac{\mu(k)^2}{\varphi(k)}\Big)\le\Lambda(N,Q)\sum_{n=M+1}^{M+N}|a_n|^2.$$

(e) Show that if the a_n are as in (b) then

$$\sum_{q\le Q}\big(\log Q/q\big)\sum_{\chi}{}^{*}|S(\chi)|^2\le\Lambda(N,Q)\sum_{n=M+1}^{M+N}|a_n|^2.$$

(f) Let $\mathcal{N}$ be the set of those integers $n\in[M+1,M+N]$ such that $(n,q)=1$ for all $q\le Q$. Put $Z=\mathrm{card}\mathcal{N}$. Show that

$$Z^2\log Q+\sum_{1<q\le Q}\big(\log Q/q\big)\sum_{\chi}{}^{*}\Big|\sum_{n\in\mathcal{N}}\chi(n)\Big|^2\le\Lambda(N,Q)Z.$$