571 Analytic number Theory I, Fall Term 2007, Solutions 14

Throughout, n is a large integer, $N_k = \lfloor n^{1/k} \rfloor$, $f_k(\alpha) = \sum_{x \le N_k} e(\alpha x^k)$, and $R(n;s,t)$ denotes the number of solutions in positive integers $x_1, \dots, x_s, y_1, \dots, y_t$ of $x_1^2 + \cdots + x_s^2 + y_1^3 + \cdots + y_t^3 = n$. (1). Also we define $\nu = \frac{1}{100}$, $\mathfrak{M}(q,a) = \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \le N_3^{\nu-3} \right\}$ and $\mathfrak{M}$ denotes the union of all $\mathfrak{M}(\mathfrak{q},\mathfrak{a})$ with $1 \le a \le q \le N_3^\nu$, $(a,q) = 1$, and $\mathfrak{m} = \left(N_3^{\nu-3}, 1 + N_3^{\nu-3}\right] \setminus \mathfrak{M}$. Let $S_k(q,a) = \sum_{x=1}^q e(ax^k/q)$, $v_k(\beta) = \sum_{m=1}^n \frac{1}{k} m^{\frac{1}{k}-1} e(m\beta)$, $V_k(\alpha,q,a) = q^{-1} S_k(q,a) v_k(\alpha - a/q)$ and define $V_k(\alpha)$ on $\mathfrak{M}$ by taking $V_k(\alpha)$ to be $V_k(\alpha,q,a)$ on $\mathfrak{M}(\mathfrak{q},\mathfrak{a})$. s,t is always one of the pairs $1,7$; $2,5$; $3,3$.

1. Show that $R(n;s,t) = \int_0^1 f_2(\alpha)^s f_3(\alpha)^t e(-\alpha n) d\alpha$.
RHS $= \sum_{\overline{x},\overline{y}} \int_0^1 e(\alpha(x_1^2 + \cdots + x_s^2 + y_1^3 + \cdots + y_t^3 - n)) d\alpha$.

2. Show that (i) $\int_0^1 \left| f_2(\alpha) f_3(\alpha)^6 \right| d\alpha \ll n^{\frac{3}{2}+\varepsilon}$, (ii) $\int_0^1 \left| f_2(\alpha)^2 f_3(\alpha)^4 \right| d\alpha \ll n^{\frac{4}{3}+\varepsilon}$, (iii) $\int_0^1 \left| f_2(\alpha)^3 f_3(\alpha)^2 \right| d\alpha \ll n^{\frac{7}{6}+\varepsilon}$.

(i) LHS $\le \left(\int_0^1 \left| f_2(\alpha)^4 \right| d\alpha \right)^{\frac{1}{4}} \left(\int_0^1 \left| f_3(\alpha)^8 \right| d\alpha \right)^{\frac{3}{4}} \ll (n^{1+\varepsilon})^{\frac{1}{4}} (n^{5/3+\varepsilon})^{\frac{3}{4}} = n^{\frac{3}{2}+\varepsilon}$. (ii) LHS $\le \left(\int_0^1 \left| f_2(\alpha)^4 \right| d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 \left| f_3(\alpha)^8 \right| d\alpha \right)^{\frac{1}{2}} \ll (n^{1+\varepsilon})^{\frac{1}{2}} (n^{5/3+\varepsilon})^{\frac{1}{2}} = n^{\frac{4}{3}+\varepsilon}$. (iii) LHS $\le \left(\int_0^1 \left| f_2(\alpha)^4 \right| d\alpha \right)^{\frac{3}{4}} \left(\int_0^1 \left| f_3(\alpha)^8 \right| d\alpha \right)^{\frac{1}{4}} \ll (n^{1+\varepsilon})^{\frac{3}{4}} (n^{5/3+\varepsilon})^{\frac{1}{4}} = n^{\frac{7}{6}+\varepsilon}$.

3. Show that there is a positive number δ such that (i) $\int_{\mathfrak{m}} \left| f_2(\alpha) f_3(\alpha)^7 \right| d\alpha \ll n^{\frac{11}{6}-\delta}$, (ii) $\int_{\mathfrak{m}} \left| f_2(\alpha)^2 f_3(\alpha)^5 \right| d\alpha \ll n^{\frac{5}{3}-\delta}$, (iii) $\int_{\mathfrak{m}} \left| f_2(\alpha)^3 f_3(\alpha)^3 \right| d\alpha \ll n^{\frac{3}{2}-\delta}$.

By the proof of Thm. 2.1 with $k = 3$, $\sup_{\mathfrak{m}} |f_3(\alpha)| \ll n^{\frac{1}{3}-2\delta}$, so $\int_{\mathfrak{m}} |f_2(\alpha)^s f_3(\alpha)^t| d\alpha \le \sup_{\mathfrak{m}} |f_3(\alpha)| \int_0^1 \left| f_2(\alpha)^s f_3(\alpha)^{t-1} \right| d\alpha$ and we can appeal to question 2.

4. Suppose that $1 \le a \le q \le N_3^\nu$, $(q,a) = 1$, $\alpha \in \mathfrak{M}(q,a)$. (i) Show that $f_k(\alpha) - V_k(\alpha,q,a) \ll n^\nu$. (ii) Show that $f_2(\alpha)^s f_3(\alpha)^t - V_2(\alpha,q,a)^s V_3(\alpha,q,a)^t \ll n^{\frac{s}{2}+\frac{t}{3}-\frac{1}{3}+\nu}$.

Note that the $\mathfrak{M}(q,a)$ defined in class depends on k. Call it $\mathfrak{M}_k(q,a)$. As $N_2^{\nu-2} \ge N_3^{\nu-3}$ it follows $\mathfrak{M}(q,a) = \mathfrak{M}_3(q,a) \subset \mathfrak{M}_2(q,a)$. Since $N_k^{2\nu} \le n^\nu$ (i) follows from Lemma 2.7. (ii) By (i) and the bounds $f_k(\alpha) \ll n^{1/k}$, $V_k(\alpha,q,a) \ll n^{1/k}$ we have $f_2^s f_3^t - V_2^s V_3^t = (f_2^s - V_2^s) f_3^t + V_2^s (f_3^t - V_3^t) \ll n^{\frac{s-1}{2}+\nu+\frac{t}{3}} + n^{\frac{s}{2}+\frac{t-1}{3}+\nu}$.

5. Prove that (i) $\int_{\mathfrak{M}} |f_2(\alpha)^s f_3(\alpha)^t - V_2(\alpha)^s V_3(\alpha)^t| d\alpha \ll n^{\frac{s}{2}+\frac{t}{3}-1-\delta}$, (ii) $R(n;s,t) = \int_{\mathfrak{M}} V_2(\alpha)^s V_3(\alpha)^t e(-\alpha n) d\alpha + O\left(n^{\frac{s}{2}+\frac{t}{3}-1-\delta} \right)$.

(i) By question 4 (ii), LHS $\ll \sum_{q \le N_3^{\nu-3}} q N_3^{\nu-3} n^{\frac{s}{2}+\frac{t}{3}-\frac{1}{3}+\nu} \ll n^{\frac{s}{2}+\frac{t}{3}-1-\frac{1}{3}+2\nu}$. (ii) By (i) and the fact that the $\mathfrak{M}(q,a)$ are disjoint, $R(n) = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right) f_2(\alpha)^s f_3(\alpha)^t e(-\alpha n) d\alpha$. Thus by (i) and question 3 this is $\int_{\mathfrak{M}} V_2(\alpha)^s V_3(\alpha)^t e(-\alpha n) d\alpha + O\left(n^{\frac{s}{2}+\frac{t}{3}-1-\delta} \right)$.

By arguments similar to those in class it can be shown that the above is

$$\frac{\Gamma(3/2)^s \Gamma(4/3)^t}{\Gamma(s/2+t/3)} \mathfrak{S}(n) n^{\frac{s}{2}+\frac{t}{3}-1} + O\left(n^{\frac{s}{2}+\frac{t}{3}-1-\delta} \right)$$

$$\text{where} \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S_2(q,a)}{q} \right)^s \left(\frac{S_3(q,a)}{q} \right)^t e(-an/q)$$

satisfies $1 \ll \mathfrak{S}(n) \ll 1$.