

571 Analytic number Theory I, Fall Term 2007, Problems 14

Return by Monday 10th December

Throughout, n is a large integer, $N_k = \lfloor n^{1/k} \rfloor$, $f_k(\alpha) = \sum_{x \leq N_k} e(\alpha x^k)$, and $R(n; s, t)$ denotes the number of solutions in positive integers $x_1, \dots, x_s, y_1, \dots, y_t$ of

$$x_1^2 + \dots + x_s^2 + y_1^3 + \dots + y_t^3 = n. \quad (1)$$

Also we define $\nu = \frac{1}{100}$, $\mathfrak{M}(q, a) = \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \leq N_3^{\nu-3} \right\}$ and \mathfrak{M} denotes the union of all $\mathfrak{M}(q, a)$ with $1 \leq a \leq q \leq N_3^\nu$, $(a, q) = 1$, and $\mathfrak{m} = (N_3^{\nu-3}, 1 + N_3^{\nu-3}] \setminus \mathfrak{M}$. Let $S_k(q, a) = \sum_{x=1}^q e(ax^k/q)$, $v_k(\beta) = \sum_{m=1}^n \frac{1}{k} m^{\frac{1}{k}-1} e(m\beta)$,

$$V_k(\alpha, q, a) = q^{-1} S_k(q, a) v_k(\alpha - a/q)$$

and define $V_k(\alpha)$ on \mathfrak{M} by taking $V_k(\alpha)$ to be $V_k(\alpha, q, a)$ on $\mathfrak{M}(q, a)$. s, t is always one of the pairs 1, 7; 2, 5; 3, 3

1. Show that

$$R(n; s, t) = \int_0^1 f_2(\alpha)^s f_3(\alpha)^t e(-\alpha n) d\alpha.$$

2. Show that (i) $\int_0^1 |f_2(\alpha) f_3(\alpha)^6| d\alpha \ll n^{\frac{3}{2}+\varepsilon}$,

$$(ii) \int_0^1 |f_2(\alpha)^2 f_3(\alpha)^4| d\alpha \ll n^{\frac{4}{3}+\varepsilon}, \quad (iii) \int_0^1 |f_2(\alpha)^3 f_3(\alpha)^2| d\alpha \ll n^{\frac{7}{6}+\varepsilon}.$$

Hint: Combine Hölder's inequality and Hua's lemma.

3. Show that there is a positive number δ such that

$$(i) \int_{\mathfrak{m}} |f_2(\alpha) f_3(\alpha)^7| d\alpha \ll n^{\frac{11}{6}-\delta}, \quad (ii) \int_{\mathfrak{m}} |f_2(\alpha)^2 f_3(\alpha)^5| d\alpha \ll n^{\frac{5}{3}-\delta},$$

$$(iii) \int_{\mathfrak{m}} |f_2(\alpha)^3 f_3(\alpha)^3| d\alpha \ll n^{\frac{3}{2}-\delta}.$$

Hint: Dirichlet's theorem on diophantine approximation and Weyl's inequality are relevant.

4. Suppose that $1 \leq a \leq q \leq N_3^\nu$, $(q, a) = 1$, $\alpha \in \mathfrak{M}(q, a)$.

(i) Show that $f_k(\alpha) - V_k(\alpha, q, a) \ll n^\nu$.

(ii) Show that $f_2(\alpha)^s f_3(\alpha)^t - V_2(\alpha, q, a)^s V_3(\alpha, q, a)^t \ll n^{\frac{s}{2} + \frac{t}{3} - \frac{1}{3} + \nu}$.

Hint: Observe that $f_2^s f_3^t - V_2^s V_3^t = (f_2^s - V_2^s) f_3^t + V_2^s (f_3^t - V_3^t)$.

5. Prove that

$$(i) \int_{\mathfrak{M}} |f_2(\alpha)^s f_3(\alpha)^t - V_2(\alpha)^s V_3(\alpha)^t| d\alpha \ll n^{\frac{s}{2} + \frac{t}{3} - 1 - \delta},$$

$$(ii) R(n; s, t) = \int_{\mathfrak{M}} V_2(\alpha)^s V_3(\alpha)^t e(-\alpha n) d\alpha + O\left(n^{\frac{s}{2} + \frac{t}{3} - 1 - \delta}\right).$$