

## Return by Monday 3rd December

1. Throughout  $k \geq 2$  and  $k$  is fixed. A  $k$ -free number is a natural number which is not divisible by the  $k$ -th power of any prime.

(i) Let  $\mu_k(n)$  denote the characteristic function of the  $k$ -free numbers. Show that  $\mu_k(n) = \sum_{m:m^k|n} \mu(m)$ .

(ii) Let  $R_k(n)$  denote the number of representations of the natural number  $n$  as the sum of a prime and a  $k$ -free number. Prove that

$$R_k(n) = \sum_{p < n} \mu_k(n-p) = \sum_{m \leq n^{1/k}} \mu(m) \pi(n-1; m^k, n).$$

(iii) Prove that if  $(n, m) > 1$ , then  $\pi(n-1, m^k, n) \leq 1$ . Also note that  $\pi(n, m^k, n) - \pi(n-1, m^k, n) \leq 1$  and deduce that

$$R_k(n) = \sum_{\substack{m \leq n^{1/k} \\ (m, n) = 1}} \mu(m) \pi(n; m^k, n) + O(n^{1/k}).$$

(iv) Let  $1 \leq y \leq n^{1/k}$ . Prove that  $\sum_{y < m \leq n^{1/k}} \pi(n; m^k, n) \ll ny^{1-k}$  and hence that

$$R_k(n) = \sum_{\substack{m \leq y \\ (m, n) = 1}} \mu(m) \pi(n; m^k, n) + O(n^{1/k} + ny^{1-k}).$$

(v) Let  $A$  be any fixed positive number. Let  $y = (\log n)^A$  and assume  $n > n_0(A, k)$  so that  $1 < y \leq n^{1/k}$ . Prove that (the Siegel–Walfisz theorem is useful here).

$$\sum_{\substack{m \leq y \\ (m, n) = 1}} \mu(m) \pi(n; m^k, n) = \text{li}(n) \sum_{\substack{m \leq y \\ (m, n) = 1}} \frac{\mu(m)}{\phi(m^k)} + O(ny \exp(-c\sqrt{\log n})).$$

(vi) Prove that  $\phi(m^k) = m^{k-1} \phi(m)$  and that  $\sum_{m > y} \frac{1}{\phi(m^k)} \ll (\log \log y) y^{1-k}$  (Theorem 2.2.9 is useful here). The  $\log \log$  could be saved with a bit of work.

(vii) Deduce that  $R_k(n) = \text{li}(n) \mathfrak{S}_k(n) + O(n(\log n)^{-A})$  where

$$\mathfrak{S}_k(n) = \sum_{\substack{m=1 \\ (m, n) = 1}}^{\infty} \frac{\mu(m)}{m^{k-1} \phi(m)}.$$

(viii) Prove that

$$\mathfrak{S}_k(n) = \prod_{p \nmid n} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right).$$