571 ANALYTIC NUMBER THEORY
I, FALL 2007, PROBLEMS 12

Return by Monday 26th December

1. Let f satisfy the hypotheses of Lemma 11 (the Hadamard product formula for entire functions of order 1), and suppose that

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|} < \infty.$$

(a) Show that there are numbers A and B and a non-negative integer K such that $f(z) = z^K e^{A+Bz} g(z)$ where $g(z) = \prod_{k=1}^{\infty}(1 - z/z_k)$.
(b) Observe that for any complex number w, $|1 - w| \le e^{|w|}$ and show that there is a number C such that $|g(z)| \le e^{C|z|}$.
(c) Prove that there is a positive constant c and arbitrarily large s such that $|\xi(s)| > \exp(c|s| \log |s|)$ [Stirling's formula is relevant].
(d) Deduce that $\sum_\rho 1/|\rho| = \infty$ where the sum is over all nontrivial zeros of the zeta function.

2. (a) Let B be the constant given in (10.30), and assume that $B = -\frac{C_0}{2} - 1 + \frac{1}{2}\log 4\pi = -0.0230957$. Show that if $\rho = 1/2 + i\gamma$ is a zero of the zeta function on the critical line, then

$$|\gamma| \ge (-1/B - 1/4)^{1/2} = 6.5611\ldots.$$

(b) Let γ be given, and put $f(\beta) = \beta/(\beta^2 + \gamma^2)$. Show that if $0 \le \beta \le 1$ then $f(\beta) \ge \beta/(1 + \gamma^2)$. Deduce that if $0 \le \beta \le 1$ then $f(\beta) + f(1 - \beta) \ge f(0) + f(1)$.
(c) Show that if $\rho = \beta + i\gamma$ is a nontrivial zero of the zeta function with $\beta \ne 1/2$ then

$$|\gamma| \ge (-2/B - 1)^{1/2} = 9.2518\ldots.$$