571 ANALYTIC NUMBER THEORY I, FALL 2007, PROBLEMS 11

Return by Monday 12th November

1. (Heilbronn) Let $Q \in \mathbb{R}$, $Q \geq 2$ and define

$$S = \sum_{p \leq Q} \sum_{q \leq Q} \left(\frac{q}{p} \right)_L$$

where the sum is over odd primes p and q.
(i) Prove that $|S|^2 \leq \pi(Q) T$ where

$$T = \sum_{q_1 \leq Q} \sum_{q_2 \leq Q} \sum_{p \leq Q} \left(\frac{q_1 q_2}{p} \right)_L .$$

(ii) Prove that

$$T^2 \leq 2\pi(Q)^2 \sum_{p_1 \leq Q} \sum_{\substack{p_2 \leq Q \\ p_2 \neq p_1}} \sum_{m \leq Q^2} \left(\frac{m}{p_1 p_2} \right)_J + 2\pi(Q)^3 Q^2.$$

(iii) Prove that

$$T^2 \ll Q^5 (\log Q)^{-3}$$

and that

$$S \ll Q^{7/4} (\log Q)^{-5/4}.$$

Can this be improved?

2. Suppose that α is real, that $\Re z > 0$ and that χ is a primitive character (mod q).
(a) Show that

$$\sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi (n+\alpha)^2 z/q} = \frac{\tau(\chi)}{q^{1/2}} z^{-1/2} \sum_{k=-\infty}^{\infty} \overline{\chi}(k) e(k\alpha/q) e^{-\pi k^2/(qz)}.$$

(b) By differentiating with respect to α, or otherwise, show that

$$\sum_{n=-\infty}^{\infty} \chi(n)(n+\alpha) e^{-\pi (n+\alpha)^2 z/q} = \frac{\tau(\chi)}{i q^{1/2}} z^{-3/2} \sum_{k=-\infty}^{\infty} \overline{\chi}(k) k e(k\alpha/q) e^{-\pi k^2/(qz)}.$$