571 ANALYTIC NUMBER THEORY I, FALL 2007, PROBLEMS 7

Return by Monday 15th October

1. Let

$$f(\theta) = 5 + 8\cos\theta + 4\cos 2\theta + \cos 3\theta.$$

Prove that for every θ, $f(\theta) \geq 0$. Hint: Try factorising $z^6 + 4z^5 + 8z^4 + 10z^3 + 8z^2 + 4z + 1$.

2. Suppose that $a_0, a_1, \dots, a_n$ are non-negative real numbers with $a_1 > a_0$ and the property that

$$f(\theta) = \sum_{m=0}^{n} a_m \cos(m\theta)$$

satisfies $f(\theta) \geq 0$. Let $\rho = \beta + i\gamma$ denote any non-trivial zero of $\zeta(s)$ with $|\gamma|$ sufficiently large. Assume only Corollary 1.17 and Lemma 6.4. The object of the question is to investigate the extent to which the zero free region could be improved with different cosine polynomials

(i) Suppose that $1 < \sigma < 2$. Show that there is an absolute constant C such that

$$\frac{a_1}{\sigma - \beta} - \frac{a_0}{\sigma - 1} \leq (a_0 + a_1 + \cdots + a_n) C \log(2 + |\gamma|).$$

(ii) Show that if $\beta \geq 2 - \sqrt{a_1/a_0}$, then

$$V(f) \leq C(1 - \beta)\log(2 + |\gamma|)$$

where

$$V(f) = \frac{\left(\sqrt{a_1} - \sqrt{a_0}\right)^2}{a_0 + a_1 + \cdots + a_n}.$$

(iii) Show that if

$$f_1(\theta) = 3 + 4\cos\theta + \cos 2\theta$$

and

$$f_2(\theta) = 5 + 8\cos\theta + 4\cos 2\theta + \cos 3\theta,$$

then $V(f_1) = \frac{7-4\sqrt{3}}{8} = 0.008974596\dots$, but $V(f_2) = \frac{13-4\sqrt{10}}{18} = 0.019493853\dots$.

As far as I know, the best polynomial of this kind so far discovered is $f_3(\theta) = (9 + 10\cos\theta)^2(3 + 10\cos\theta)^2$ (Stechkin 1970), which has large coefficients, but leads to $V(f_3) = 0.028646482\dots$. It is also known (French 1966) that $V(f) < \frac{4}{65} = 0.030769231\dots$ for any non-negative cosine polynomial f. The polynomial f_2 is due to Landau, 1909.