571 ANALYTIC NUMBER THEORY I, FALL 2007, PROBLEMS 6

Return by Monday 8th October

1. Let χ be a Dirichlet character (mod q), and let k denote the order of χ in the character group.
(a) Show that if $(a, q) = 1$ then $\chi(a)$ is a k^{th} root of unity.
(b) Show that each k^{th} root of unity occurs precisely $\varphi(q)/k$ times among the numbers $\chi(a)$ as a runs over the $\varphi(q)$ reduced residue classes (mod q).

2. Let $k \in \mathbb{N}$ and p be a prime number. Let $l = (k, p-1)$ and let $\mathcal{A}_k$ denote the set of non-principal characters modulo p whose order divides l. Let $N(a)$ denote the number of solutions of the congruence $x^k \equiv a \pmod{p}$. Show that $\text{card}\mathcal{A}_k = l - 1$, $N(0) = 1$, $N(a) = l$ or 0 when $p \nmid a$, and

$$1 + \sum_{\chi \in \mathcal{A}_k} \chi(a) = N(a).$$

3. (a) Show that if $y \geq 0$ then

$$-\frac{\pi}{2} = \text{si}(0) \leq \text{si}(y) \leq \text{si}(\pi) = 0.28114\ldots .$$

(b) Show that if $y > 0$ then

$$\Im \int_y^\infty \frac{e^{iu}}{u}\, du = \Im \int_y^{y+i\infty} \frac{e^{iz}}{z}\, dz.$$

(c) Deduce that if $y > 0$ then $|\,\text{si}(y)| < 1/y$.