571 ANALYTIC NUMBER THEORY I, FALL 2007, PROBLEMS 5

Return by Monday 1st October

1. Show that for arbitrary real or complex numbers $c_1, \dots, c_q$,

$$\sum_{\chi} \Big| \sum_{n=1}^{q} c_n \chi(n) \Big|^2 = \varphi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2$$

where the sum on the left hand side runs over all Dirichlet characters $\chi \pmod q$.

2. Show that for arbitrary real or complex numbers c_χ,

$$\sum_{n=1}^{q} \Big| \sum_{\chi} c_\chi \chi(n) \Big|^2 = \varphi(q) \sum_{\chi} |c_\chi|^2$$

where sums over χ is extended over all Dirichlet characters $\pmod q$.

3. (Mertens (1895a,b)) Let $r(n) = \sum_{d|n} \chi(d)$.

(a) Show that if χ is a non-principal character $\pmod q$, then

$$\sum_{n>x} \frac{\chi(n)}{\sqrt{n}} \ll_\chi \frac{1}{\sqrt{x}}.$$

(b) Show that if χ is a non-principal character $\pmod q$, then

$$\sum_{n \le x} \frac{r(n)}{n^{1/2}} = 2x^{1/2} L(1,\chi) + O_\chi(1).$$

(c) Recall that if χ is quadratic then $r(n) \ge 0$ for all n, and that $r(n^2) \ge 1$. Deduce that if χ is a quadratic character, then the left hand side above is $\gg \log x$.

(d) Conclude that if χ is a quadratic character, then $L(1,\chi) > 0$.