571 ANALYTIC NUMBER THEORY I, FALL 2007, PROBLEMS 4

Return by Monday 24th September

1. Let $T(x) = \sum_{n \le x} \log n$. Show that $\sum_{n \le x} \psi(x/n) = T(x) = x \log x + O(x)$.

2. New question needed.

3. Suppose throughout that $0 < \alpha \le 1$. For $\sigma > 1$ we define the *Hurwitz zeta function* by the formula

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}.$$

Thus $\zeta(s, 1) = \zeta(s)$.
(a) Show that $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$.
(b) Show that if $x \ge 0$ then

$$\zeta(s, \alpha) = \sum_{0 \le n \le x} (n + \alpha)^{-s} + \frac{(x + \alpha)^{1-s}}{s - 1} + \frac{\{x\}}{(x + \alpha)^s} - s \int_x^{\infty} \{u\}(u + \alpha)^{-s-1}\, du.$$

(c) Deduce that $\zeta(s, \alpha)$ is an analytic function of s for $\sigma > 0$ apart from a simple pole at $s = 1$ with residue 1.
(d) Show that

$$\lim_{s \to 1} \Big(\zeta(s, \alpha) - \frac{1}{s - 1} \Big) = 1/\alpha - \log \alpha - \int_0^{\infty} \frac{\{u\}}{(u + \alpha)^2}\, du.$$

4. Let $t(n) = (-1)^{\Omega(n) - \omega(n)} \prod_{p|n} (p - 1)^{-1}$, and put $T(s) = \sum_n t(n) n^{-s}$.
(a) Show that for $\sigma > 0$, $T(s)$ has the absolutely convergent Euler product

$$T(s) = \prod_p \Big(1 + \frac{1}{(p - 1)(p^s + 1)} \Big).$$

(b) Determine all zeros of the function $1 + 1/((p - 1)(p^s + 1))$.
(c) Show that the line $\sigma = 0$ is a natural boundary of the function $T(s)$.