

**MATH 571 ANALYTIC NUMBER
THEORY I FALL 2007, PROBLEMS 3**

To be submitted by Monday 17th September

1. Let $A_k(x) = \sum_{n \leq x} a_n (\log n)^k$. (a) Show that

$$A_0(x) - \frac{A_1(x)}{\log x} = a_1 + \int_2^x \frac{A_1(u)}{u(\log u)^2} du.$$

(b) Suppose that $A_1(x) \ll x^\theta$ where $\theta > 0$. Show that

$$A_0(x) = \frac{A_1(x)}{\log x} + O(x^\theta (\log x)^{-2}).$$

(c) Let σ_c denote the abscissa of convergence of $\sum a_n n^{-s}$, and σ'_c the abscissa of convergence of $\sum a_n (\log n) n^{-s}$. Show that $\sigma'_c = \sigma_c$.

2. (a) Show that if $\alpha(s) = \sum a_n n^{-s}$ has abscissa of convergence $\sigma_c < \infty$, then $\lim_{\sigma \rightarrow \infty} \alpha(\sigma) = a_1$.

(b) Show that $\zeta'(s) = -\sum_{n=1}^{\infty} (\log n) n^{-s}$ for $\sigma > 1$.

(c) Show that $\lim_{\sigma \rightarrow \infty} \zeta'(\sigma) = 0$.

(d) Show that there is no halfplane in which $1/\zeta'(s)$ can be written as a convergent Dirichlet series.

3. (a) Show that if a_n is totally multiplicative, and if $\alpha(s) = \sum a_n n^{-s}$ has abscissa of convergence σ_c , then for $\sigma > \sigma_c$

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n n^{-s} = (1 - 2a_2 2^{-s}) \alpha(s).$$

(b) Show that when $\sigma > 1$

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1 - 2^{1-s}) \zeta(s).$$

(c) Show that the series on the left above has abscissa of convergence 0.