MATH 571 ANALYTIC NUMBER THEORY I, FALL 2007, PROBLEMS 1

Due 5th September

Revision Problems

1. Solve where possible
   (i) $77x \equiv 84 \pmod{143}$;
   (ii) $77x \equiv 84 \pmod{147}$;

2. Find the last digit of 7^{2001} and of 13^{143}.

3. Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is a totally multiplicative function with $f(n) = 0$ or ± 1. Prove that

$$\sum_{m|n} f(m) \geq 0 \quad \text{and} \quad \sum_{m|n^2} f(m) \geq 1.$$

4. Suppose that $0 \leq a < m$ and $\mathcal{A}$ denotes the non-negative integers in the residue class $a \pmod{m}$. Show that if $z \in \mathbb{C}$ and $|z| < 1$, then

$$\sum_{n \in \mathcal{A}} z^n = \frac{z^a}{1 - z^m}.$$

5. (L. Mirsky and D. J. Newman) Suppose that $K \geq 2$, $0 \leq a_k < m_k$ for $1 \leq k \leq K$ and that $m_1 < m_2 < \ldots < m_K$. This is called a *family of covering congruences* when ever integer x satisfies at least one of the congruences $x \equiv a_k \pmod{m_k}$. A system of covering congruences is called *exact* when for every value of x there is exactly one value of k such that $x \equiv a_k \pmod{m_k}$. Show that if the system is exact, then

$$\sum_{k=1}^{K} \frac{z^{a_k}}{1 - z^{m_k}} = \frac{1}{1 - z}.$$

When $z = re(1/m_K)$ (where $e(z)$ denotes $e^{2\pi i z}$) with $r \in \mathbb{R}_{>0}$ and $r \to 1-$, show that the left hand side above is

$$\sim \frac{e(a_K/m_K)}{m_K(1 - r)}$$

whereas the right hand side is bounded for z in a neighbourhood of $e(1/m_K)$.