

568 NUMBER THEORY II, SPRING TERM 2007, PROBLEMS 13

Return by Tuesday 1st May

1. Let $E_k(z) = G_k(z)/(2\zeta(2k))$, $q = e^{2\pi iz}$. Show that

$$E_2(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$E_3(z) = 1 - 540 \sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

$$E_4(z) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n,$$

$$E_5(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n,$$

$$E_6(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n,$$

2. Prove that $\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{\infty} \sigma_3(m)\sigma_3(n-m)$.

3. Prove that $11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m)$.

4. Prove that $756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 691.252 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m)$. Deduce Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.