

568 NUMBER THEORY II, SPRING TERM 2007, PROBLEMS 11

Return by Tuesday 17th April

Γ denotes the modular group and S, T are its generators, $S(z) = -1/z$, $T(z) = z + 1$. Given a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with real coefficients, $d = d_Q = b^2 - 4ac$ is called the discriminant of Q .

1. (i) Find all elements A of Γ which commute with S .
(ii) Find all elements A of Γ which commute with T .
(iii) Find the smallest $n > 0$ such that $(ST)^n = I$.
(iv) Determine all A in Γ which leave i fixed.
(v) Determine all A in Γ which leave $\rho = e(1/3)$ fixed.
2. Prove that if $A \in \Gamma$, and $(x, y)^T = A(x', y')^T$, then the quadratic form Q' defined by $Q'(x', y') = Q(x, y)$ satisfies $d_{Q'} = d_Q$. Two forms related in this way are called equivalent. This relation separates all forms into equivalence classes. The forms in the same class have the same discriminant and the ranges $Q(\mathbb{Z}^2)$ coincide.

In the remaining exercises it will be supposed that the quadratic forms have positive coefficients of x^2 and y^2 and negative discriminant. The associated polynomial $Q(z, 1)$ has two complex roots. The one in \mathbb{H} is called the representative of Q .

3. (i) If d is fixed, prove that there is a bijection between the set of forms with discriminant d and the members of \mathbb{H} .
(ii) Prove that two quadratic forms with discriminant d are equivalent iff their representatives are equivalent under Γ .

A reduced form is one whose representative lies in the fundamental domain \mathbb{D} , the set of z such that either $|z| > 1$ and $-1/2 \leq \Re z < 1/2$ or $|z| = 1$ and $-1/2 \leq \Re z \leq 0$. Thus two reduced forms are equivalent iff they are identical, and moreover each equivalence class contains exactly one reduced form.

4. Prove that $Q(x, y) = ax^2 + bxy + cy^2$ is reduced iff either $-a < b \leq a < c$ or $0 \leq b \leq a = c$.

In questions 5,6 it is assumed that the quadratic forms have integer coefficients.

5. Prove that the number of reduced forms with a given discriminant $d < 0$ is finite. The number of such classes is called the class number and is denoted by $h(d)$.
6. When $d = -3, -4, -7, -8, -11, -15, -19, -20, -23$ determine all reduced forms with discriminant d , and the corresponding class number $h(d)$.