MATH 568 INTRODUCTION TO NUMBER THEORY II, SPRING TERM 2007, PROBLEMS 9

Return by Tuesday 3rd April

1. Let $c_q(n)$ denote the Ramanujan sum

$$c_q(n) = \sum_{\substack{r=1\\(r,q)=1}}^{q} e(nr/q).$$

(i) Prove that $c_q(n) = \sum_{r|(q,n)} r\mu(q/r)$.
(ii) Prove that for each fixed n, $c_q(n)$ is a multiplicative function of q.
(iii) Prove that

$$c_{p^k}(n) = \begin{cases} \phi(p^k) & p^k|n, \\ -p^{k-1} & p^{k-1}\|n, \\ 0 & p^{k-1}\nmid n. \end{cases}$$

(iv) Prove that $c_q(n) = \phi(q)\frac{\mu(q/(q,n))}{\phi(q/(q,n))}$.

2. Prove that if $(a,q)=1$, then, as $\sigma \to 1+$,

$$\sum_p e(ap/q)\frac{\log p}{p^\sigma} = \frac{\mu(q)}{\phi(q)(\sigma-1)} + O(1).$$

3. (i) Suppose that $n \equiv 2$ or $6 \pmod 8$. Prove that there is an prime p such that $p \equiv -1 \pmod n$ and

$$\left(\frac{-(p+1)/n}{p}\right)_L = 1.$$

Hint: Consider $p \equiv n-1 \pmod 4n$.
(ii) Given n and p as above show that there are integers a_{11}, a_{12} such that $a_{11} > 0$, $a_{11}p - a_{12}^2 > 0$ and

$$\det \begin{bmatrix} a_{11} & a_{12} & 1 \\ a_{12} & p & 0 \\ 1 & 0 & n \end{bmatrix} = 1.$$

This is the crucial ingredient in Legendre's "proof" that every such n is the sum of three squares. The point is that all forms with determinantt 1 turn out to be equivalent to $x_1^2+x_2^2+x_3^2$ and so it suffices to find any form with that determinant which reperesents n. The case $n \equiv 1$, 3 or 5 (mod 8) is similar but a little more complicated.