MATH 568 INTRODUCTION TO NUMBER THEORY II, SPRING TERM 2007, PROBLEMS 7

Return by Tuesday 20th March

1. Let $\mathcal{A}$ be the set of integers composed entirely of primes $p \le A_1$, and let $\mathcal{B}$ be the set of integers composed entirely of primes $p > A_1$. Then n is uniquely of the form $n = ab$, $a \in \mathcal{A}$, $b \in \mathcal{B}$. Let $\delta(A_1, A_2)$ denote the density of those n such that $a \le A_2$.
(a) Give a formula for $\delta(A_1, A_2)$.
(b) Show that $\delta(A_1, A_2) \gg (\log A_2)/\log A_1$ for $2 \le A_2 \le A_1$.

2. Let $a_n = 1 + \cos\log n$, and note that $a_n \ge 0$ for all n.
(a) Show that

$$\sum_{n=1}^{\infty} a_n n^{-s} = \zeta(s) + \frac{1}{2}\zeta(s+i) + \frac{1}{2}\zeta(s-i)$$

for $\sigma > 1$.
(b) By Corollary 1.15, or otherwise, show that

$$\sum_{n\le x} \frac{a_n}{n} = \log x + O(1).$$

(c) Show that

$$\sum_{n\le x} a_n = \Big(1 + \frac{x^i}{2(1+i)} + \frac{x^{-i}}{2(1-i)}\Big)x + O(\log x).$$

(d) Deduce that

$$\liminf_{x\to\infty} \frac{1}{x}\sum_{n\le x} a_n = 1 - \frac{1}{\sqrt{2}}, \qquad \limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} a_n = 1 + \frac{1}{\sqrt{2}}.$$

Thus for the coefficients a_n we have an analogue of Mertens' estimate but not an analogue of the Prime Number Theorem.

3. Suppose that t is a fixed non-zero real number. Prove that, as $x \to \infty$,

$$\sum_{n\le x} n^{-1-it} - \frac{1}{it} - 1 + (1+it)\int_1^{\infty} \{u\}u^{-2-it}dt = -\frac{x^{-it}}{it} + O(x^{-1}).$$

It is the oscillating term, x^{-it} on the right which prevents the series from converging.