MATH 568 SPRING 2007, PROBLEMS 6

Return by Tuesday 6th March

1. Determine $\sum \varphi(n)n^{-s}$, $\sum \sigma(n)n^{-s}$, and $\sum |\mu(n)|n^{-s}$ in terms of the zeta function.

2. Let q be a positive integer. Show that if $\sigma > 1$ then

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-s} = \zeta(s) \prod_{p|q} (1 - p^{-s}).$$

3. Show that if $\sigma > 1$ then

$$\sum_{n=1}^{\infty} d(n)^2 n^{-s} = \zeta(s)^4 / \zeta(2s).$$

4. Let $\sigma_a(n) = \sum_{d|n} d^a$. Show that

$$\sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)n^{-s} = \zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b)$$

when $\sigma > \max\big(1, 1 + \Re a, 1 + \Re b, 1 + \Re(a+b)\big)$.

5. Let $F(s) = \sum_p (\log p)p^{-s}$, $G(s) = \sum_p p^{-s}$ for $\sigma > 1$. Show that in this halfplane,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{k=1}^{\infty} F(ks),$$

$$F(s) = -\sum_{d=1}^{\infty} \mu(d) \frac{\zeta'}{\zeta}(ds),$$

$$\log \zeta(s) = \sum_{k=1}^{\infty} G(ks)/k,$$

$$G(s) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log \zeta(ds).$$