MATH 568 SPRING 2007, PROBLEMS 5

Return by Tuesday 27th February

1. Show that $d(n) \le \sqrt{3n}$ with equality if and only if $n = 12$.

2. Let $k \in \mathbb{N}$, let $d_k(n)$ denote the number of $(m_1, \dots, m_k) \in \mathbb{N}^k$ such that $m_1 \dots m_k = n$ and define $S_k(x) = \sum_{n \le x} d_k(n)$, $T_k(x) = \sum_{n \le x} d_k(n)/n$.
(a) Prove that if $k \ge 2$, then $S_k(x) = xT_{k-1}(x) + O(S_{k-1}(x))$.
(b) Prove that $T_k(x) = x^{-1}S_k(x) + \int_1^x S_k(t)\frac{dt}{t^2}$.
(c) Prove that

$$T_k(x) = \frac{(\log x)^k}{k!} + O\big((\log x)^{k-1}\big)$$

and, when $k \ge 2$,

$$S_k(x) = x\frac{(\log x)^{k-1}}{(k-1)!} + O\big(x(\log x)^{k-2}\big).$$

3. (Bateman (1949)) Let $\Phi_q(z)$ denote the q^{th} cyclotomic polynomial,

$$\Phi_q(z) = \prod_{\substack{a=1 \\ (a,q)=1}}^{q} (z - e(a/q))$$

where $e(\theta) = e^{2\pi i\theta}$.
(a) Show that

$$\prod_{d|q} \Phi_d(z) = z^q - 1.$$

(b) Show that

$$\Phi_q(z) = \prod_{d|q} (z^d - 1)^{\mu(q/d)}.$$

(c) If $P(z) = \sum p_n z^n$ and $Q(z) = \sum q_n z^n$ are polynomials with real coefficients then we say that $P \preccurlyeq Q$ if $|p_n| \le q_n$ for all non-negative integers n. Show that if $P_1 \preccurlyeq Q_1$ and $P_2 \preccurlyeq Q_2$ then $P_1 + P_2 \preccurlyeq Q_1 + Q_2$ and $P_1P_2 \preccurlyeq Q_1Q_2$.
(d) Show that $\Phi_q(z) \preccurlyeq Q_q(z)$ where

$$Q_q(z) = \prod_{d|q} \big(1 + z^d + z^{2d} + \dots + z^{q-d}\big).$$

(e) Show that $Q_q(1) = q^{d(q)/2}$.
(f) Show that for any $\varepsilon > 0$ there is a $q_0(\varepsilon)$ such that if $q > a_0(\varepsilon)$ then all coefficients of Φ_q have absolute value not exceeding

$$\exp\big(q^{(\log 2+\varepsilon)/\log\log q}\big).$$

Erdős (1949) had shown that there are infinitely many q such that the q–th cyclotomic polynomial has coefficients

$$> \exp\big(q^{(\log 2-\varepsilon)/\log\log q}\big)$$

and Vaughan (1975) showed that this is true even with $\epsilon = 0$.