

**MATH 568 INTRODUCTION TO NUMBER
THEORY II, SPRING TERM 2007, PROBLEMS 4**

Return by Tuesday 20th February

1. (a) Let a_1, a_2, \dots be non-zero integers, and define $d_n = \text{lcm}[a_1, \dots, a_n]$. Given n , prove that there are integers b_1, b_2, \dots, b_n such that $\frac{1}{d_n} = \frac{b_1}{a_1} + \dots + \frac{b_n}{a_n}$.
- (b) Let $d_n = \text{lcm}[1, 2, \dots, n]$. Prove that $d_n = e^{\psi(n)}$.
- (c) Let $P \in \mathbb{Z}[x]$, $\deg P \leq n$. Put $I = I(P) = \int_0^1 P(x) dx$. Prove that $I d_{n+1} \in \mathbb{Z}$, and hence that $d_{n+1} \geq 1/|I|$ if $I \neq 0$.
- (d) Prove that there is a polynomial P as above so that $I d_{n+1} = 1$.
- (e) Prove that $\max_{0 \leq x \leq 1} |x^2(1-x)^2(2x-1)| = 5^{-5/2}$.
- (f) For $P(x) = (x^2(1-x)^2(2x-1))^{2n}$, prove that $0 < I < 5^{-5n}$.
- (g) Prove that $\psi(10n+1) \geq (\frac{1}{2} \log 5) \cdot 10n$.

2. Prove that

$$\sum_{n=1}^{\infty} \frac{n}{10^{n^2}}$$

is irrational.

3. (a) Prove that if $x \geq 1$, then

$$\sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1.$$

(b) Prove that

$$-1 + 1/x \leq \sum_{n \leq x} \frac{\mu(n)}{n} \leq 1 + 1/x.$$

In fact we know that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0,$$

but this is equivalent to the prime number theorem in the sense that it follows from the prime number theorem and there is a relatively simple proof that it implies the prime number theorem.