MATH 568 INTRODUCTION TO NUMBER THEORY II, SPRING TERM 2007, PROBLEMS 3

Return by Tuesday 6th February

1. This question investigates whether there exists an arithmetic function θ such that $\theta * \theta = \mu$ and $\theta(1) \geq 0$.

(i) Prove that θ exists and is uniquely determined.

(ii) Prove that $\theta \in \mathcal{M}$.

(iii) Prove that $\theta(p^k) = (-1)^k \binom{\frac{1}{2}}{k}$.

2.

3. Let $f(n)$ denote the number of solutions of $x^3 + y^3 = n$ in natural numbers x, y. Show that

$$\sum_{n \leq X} f(n) = AX^{2/3} + O\left(X^{1/3}\right) \quad \text{where} \quad A = \int_0^1 (1 - \alpha^3)^{1/3} d\alpha.$$

Note that $A = \frac{1}{3}B(4/3, 1/3) = \frac{\Gamma(4/3)^2}{\Gamma(5/3)} = \frac{1}{\pi}3^{3/2}\Gamma(4/3)^3$.

4. Show that the number $N(X)$ of different natural numbers of the form $2^r 3^s$ with $r \in \mathbb{N}$, $s \in \mathbb{N}$ and $2^r 3^s \leq X$ satisfies

$$N(X) = \frac{(\log X)^2}{2(\log 2)(\log 3)} + O(\log X)$$

as $X \to \infty$. Hint: Note that the condition $2^r 3^s \leq X$ is equivalent to $r \log 2 + s \log 3 \leq \log X$.