# 567 NUMBER THEORY I, FALL 2008, SOLUTIONS 12

1. Show that for arbitrary real or complex numbers $c_1, \dots, c_q$,

$$\sum_\chi \Big| \sum_{n=1}^{q} c_n \chi(n) \Big|^2 = \varphi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2$$

where the sum on the left hand side runs over all Dirichlet characters $\chi \pmod q$.

LHS= $\sum_{m=1}^q \sum_{n=1}^q c_m \overline{c}_n \sum_\chi \chi(m)\overline{\chi}(n)$ and the inner sum is 0 unless $(mn,q)=1$ and $m =\equiv n \pmod q$, in which case it is $\phi(q)$.

2. Show that for arbitrary real or complex numbers $c_\chi$, $\sum_{n=1}^q \Big| \sum_\chi c_\chi \chi(n) \Big|^2 = \varphi(q) \sum_\chi |c_\chi|^2$ where sums over $\chi$ are extended over all Dirichlet characters (mod $q$).

LHS= $\sum_{\chi_1} \sum_{\chi_2} c_{\chi_1} \overline{c}_{\chi_2} \sum_{n=1}^q \chi_1 \overline{\chi}_2(n)$ and the innermost sum is 0 unless $\chi_1 = \chi_2$, in which case it is $\phi(q)$.

3. (Mertens (1895a,b)) Let $r(n) = \sum_{d|n} \chi(d)$. (a) Show that if $\chi$ is a non-principal character (mod $q$), then $\sum_{n>x} \frac{\chi(n)}{\sqrt{n}} \ll_\chi \frac{1}{\sqrt{x}}$. (b) Show that if $\chi$ is a non-principal character (mod $q$), then $\sum_{n\le x} \frac{r(n)}{n^{1/2}} = 2x^{1/2} L(1,\chi) + O_\chi(1)$. (c) Recall that if $\chi$ is quadratic then $r(n) \ge 0$ for all $n$, and that $r(n^2) \ge 1$. Deduce that if $\chi$ is a quadratic character, then the left hand side above is $\gg \log x$. (d) Conclude that if $\chi$ is a quadratic character, then $L(1,\chi) > 0$.

(a) Let $S(y) = \sum_{n\le y} \chi(n)$. Then $\sum_{x<n\le y} \frac{\chi(n)}{\sqrt{n}} = \frac{S(X)-S(x)}{\sqrt{X}} + \int_x^X \frac{S(t)-S(x)}{2t^{3/2}} dt$. Since $S(t) \ll_\chi 1$, this is $\ll_\chi \frac{1}{\sqrt{X}} + \int_x^X \frac{1}{2t^{3/2}} dt$. (b) It is useful to observe that $\sum_{m\le y} \frac{1}{\sqrt{m}} = \frac{\lfloor y \rfloor}{\sqrt{y}} + \int_1^y \frac{\lfloor y \rfloor}{2t^{3/2}} dt = y^{1/2} - \frac{\{y\}}{\sqrt{y}} + \int_1^y \frac{1}{2t^{1/2}} dt - \int_1^\infty \frac{\{t\}}{2t^{3/2}} dt + \int_y^\infty \frac{\{t\}}{2t^{3/2}} dt = 2y^{1/2} + C_1 + O(y^{-1/2})$. Here $\{t\} = t - \lfloor t \rfloor$ and $C_1$ is a constant. Now we apply Dirichlet's method of the hyperbola to the sum in question. Let $D = \sqrt{x}$. Then $\sum_{n\le x} \frac{r(n)}{n^{1/2}} = \sum_{d\le D} \frac{\chi(d)}{\sqrt{d}} \sum_{m\le x/d} \frac{1}{\sqrt{m}} + \sum_{m\le x/D} \frac{1}{\sqrt{m}} \sum_{D<d\le x/m} \frac{\chi(d)}{\sqrt{d}}$. By (a), the second double sum here is $\ll_\chi \sum_{m\le x/D} \frac{1}{\sqrt{mD}} \ll \frac{\sqrt{x}}{D} = 1$. The first double sum is $\sum_{d\le D} \frac{\chi(d)}{\sqrt{d}} \left( 2(x/d)^{1/2} + C_1 + O(d^{1/2}x^{-1/2}) \right)$. The $O$ term here makes a total contribution to the sum of $\ll Dx^{-1/2} = 1$. By (a) with $x = 1$ the contribution from $C_1$ is $\ll_\chi 1$. Finally the main term gives $2x^{1/2} \sum_{d\le D} \frac{\chi(d)}{d}$ and by the theory of $L$–functions developed in class the sum here is $L(1,\chi) + O_\chi(1/D)$. (c) We have $\sum_{n\le x} \frac{r(n)}{n^{1/2}} \ge \sum_{m\le\sqrt{x}} \frac{1}{m} \ge \sum_{m\le\sqrt{x}} \int_m^{m+1} \frac{dt}{t} \ge \int_1^{\sqrt{x}} \frac{dt}{t} = \frac{1}{2}\log x$. (d) If $L(1,\chi) = 0$, then it would follow that $\log x \ll 1$ for all $x$!