MATH 567 INTRODUCTION TO NUMBER THEORY I, FALL TERM 2008, SOLUTIONS 11

1. (a) Prove that if $x \ge 1$, then $\sum_{n\le x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1$. (b) Prove that $-1 + 1/x \le \sum_{n\le x} \frac{\mu(n)}{n} \le 1 + 1/x$. (a) $\lfloor x/n \rfloor = \sum_{\substack{m\le x\\ n|m}} 1$, so $\displaystyle\sum_{n\le x} \mu(n)\lfloor x/n\rfloor = \sum_{m\le x}\sum_{n|m} \mu(n) = 1$.
(b) We have $x\sum_{n\le x} \frac{\mu(n)}{n} = \sum_{n\le x}\mu(n)\lfloor x/n\rfloor + \sum_{n\le x}\mu(n)\left(x/n - \lfloor x/n\rfloor\right)$. The first sum is 1 and in the second the general term is bounded by 1 in absolute value, so the sum lies between $\pm x$.

2. (a) Let $a_1, a_2, \dots$ be non-zero integers, and define $d_n = \operatorname{lcm}[a_1, \dots, a_n]$. Given n, prove that there are integers $b_1, b_2, \dots, b_n$ such that $\frac{1}{d_n} = \frac{b_1}{a_1} + \cdots + \frac{b_n}{a_n}$. (b) Let $d_n = \operatorname{lcm}[1, 2, \dots, n]$. Prove that $d_n = e^{\psi(n)}$. (c) Let $P \in \mathbb{Z}[x]$, $\deg P \le n$. Put $I = I(P) = \int_0^1 P(x)\,dx$. Prove that $Id_{n+1} \in \mathbb{Z}$, and hence that $d_{n+1} \ge 1/|I|$ if $I \ne 0$. (d) Prove that there is a polynomial P as above so that $Id_{n+1} = 1$. (e) Prove that $\max_{0\le x\le 1} |x^2(1-x)^2(2x-1)| = 5^{-5/2}$. (f) For $P(x) = \left(x^2(1-x)^2(2x-1)\right)^{2n}$, prove that $0 < I < 5^{-5n}$. (g) Prove that $\psi(10n+1) \ge (\frac{1}{2}\log 5)\cdot 10n$.

(a) Proof by induction. $n = 1$ is trivial. Assume case n. Then $d_{n+1} = [d_n, a_{n+1}] = d_n a_{n+1}/(d_n, a_{n+1})$ and there are c_i such that $(d_n, a_{n+1}) = c_2 d_n + c_1 a_{n+1}$, etc. (b) Let p be any prime number and let $k = \lfloor (\log n)/\log p\rfloor$. Then $p^k \le n$ so $p^k | d_n$, but since $p^{k+1} > n$ if $p^{k+1}|d_n$, then we could replace d_n by d_n/p. Hence $d_n = \prod_{p\le n} p^{\lfloor (\log n)/\log p\rfloor}$, etc. (c) We have $P(x) = \sum_{k=0}^n c_k x^k$, $c_k \in \mathbb{Z}$. Then $d_{n+1} I = \sum_{k=0}^n \frac{c_k d_{n+1}}{k+1} \in \mathbb{Z}$. Thus, if $I \ne 0$, then $|d_{n+1} I| \ge 1$. (d) Choose the b_k as in (a) but with n replaced by $n+1$ and let $P(x) = \sum_{k=0}^n b_{k+1} x^k$. (e) By differentiating $x^2(1-x)^2(2x-1)$ it is readily ascertained that the extremal values ocur when $x = \frac{1}{2} \pm \frac{1}{2\sqrt{5}}$. (f) By (e) $0 \le P(x) \le 5^{-5n}$ with equality only at $x = 0, \frac{1}{2}, 1$ for the lower bound and for $x = \frac{1}{2} \pm \frac{1}{2\sqrt{5}}$ for the upper bound. Since P is continuous, $0 < I < 5^{-5n}$. (g) We have $\deg P = 10n$, so by (b), $\exp\left(\psi(10n+1)\right) = d_{10n+1}$. By (c) and (f), $d_{10n+1} \ge 5^{5n}$, so $\psi(10n+1) \ge 5n\log 5$. $\frac{1}{2}\log 5 = 0.80471\dots$, so it follows that for large x, $\psi(x) > 0.8x$.

3. Prove that all the characters modulo 8 are real.

Whenever $(a, 8) = 1$ we have $a^2 \equiv 1 \pmod 8$. Hence for any character χ modulo 8 we have $\chi(a)^2 = \chi(a^2) = \chi(1) = 1$, so $\chi(a) = \pm 1$.