Math 567 Number Theory I, Solutions 9

1. $\Omega(n)$ denotes the total number of prime factors on n. Show that $\Omega(n) \le \frac{\log n}{\log 2}$. We have $2^{\Omega(n)} \le n$.

2. Show that $\sum_{n\le x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x + O(\log x)$ for $x \ge 2$. We have $\sigma(n) = \sum_{m|n} m = \sum_{m|n} n/m$. Thus $\sum_{n\le x} \sigma(n)/n = \sum_{n\le x}\sum_{m|n} 1/m = \sum_{m\le x} \frac{1}{m}\left\lfloor \frac{x}{m}\right\rfloor = x\sum_{m\le x} m^{-2} + O(\log x) = x\sum_{m=1}^{\infty} m^{-2} + O(\log x)$.

3. Let $D(x) = \sum_{n\le x} d(n)$. Show that $\sum_{n\le x} \frac{d(n)}{n} = \frac{D(x)}{x} + \int_1^x \frac{D(u)}{u^2}du$. Deduce that $\sum_{n\le x} \frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + O(\log x)$. $\sum_{n\le x} \frac{d(n)}{n} = \sum_{n\le x} d(n)\left(\frac{1}{x} + \int_n^x \frac{du}{u^2}\right) = \frac{D(x)}{x} + \int_1^x \frac{D(u)}{u^2}du$. By Dirichlet's estimate for $D(x)$ this is $x(\log x + 2\gamma - 1) + O(x^{-1/2}) + \int_1^x \frac{\log u + 2\gamma - 1}{u} + O(u^{-1/2})du = \frac{1}{2}(\log x)^2 + 2\gamma\log x + O(1) = \frac{1}{2}(\log x)^2 + O(\log x)$. Alternatively $\sum_{n\le x} \frac{d(n)}{n} = \sum_{n\le x} \frac{1}{n}\sum_{m|n} 1 = \sum_{m\le x} \frac{1}{m}\sum_{l\le x/m} \frac{1}{l} = \sum_{m\le x} \frac{1}{m}\big(\log(x/m) + O(1)\big) = \sum_{m\le x} \frac{1}{m}\int_m^x \frac{du}{u} + O(\log x) = \int_1^x \left(\sum_{m\le u} \frac{1}{m}\right)\frac{du}{u} + O(\log x) = \int_1^x \log u \frac{du}{u} + O(\log x)$.

4. A number $n \in \mathbb{N}$ is *squarefree* when it has no repeated prime factors. For $x \in \mathbb{R}$, $x \ge 1$ let $Q(x)$ denote the number of squarefree numbers not exceeding x. (i) Show that if $n \in \mathbb{N}$, then $Q(n) \ge n - \sum_p \left\lfloor \frac{n}{p^2}\right\rfloor$. (ii) Show that $\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}$. (iii) Show that $Q(n) > n/2$ for all $n \in \mathbb{N}$. (iv) Show that every integer $n > 1$ is a sum of two squarefree numbers. (i) We have $n - Q(n) = \operatorname{card}\mathcal{S}$ where $\mathcal{S}$ is the set of natural numbers $m \le n$ which are divisible by the square of some prime. The total number of natural numbers $m \le n$ which are divisible by p^2 is $\lfloor np^{-2}\rfloor$ and so $\operatorname{card}\mathcal{S} \le \sum_p \lfloor np^{-2}\rfloor$. (ii) A prime p is 2 or odd and ≥ 3, and 9 is not a prime. Hence $\sum_p p^{-2} < \frac{1}{4} + \sum_{k=1}^{\infty}(2k+1)^{-2} < \frac{1}{4} + \sum k = 1^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}$. (iii) By (i) & (ii), $Q(n) \ge n - n\sum_p p^{-2} > \frac{n}{2}$. (iv) Let $\mathcal{A}$ denote the set of squarefree $k \le n-1$ and let $\mathcal{B}$ denote the set of $l \le n-1$ with $n-l$ squarefree. Then $\mathcal{A}$, $\mathcal{B} \subset [1, n-1]$ and, by (iii), $\operatorname{card}\mathcal{A} = \operatorname{card}\mathcal{B} > \frac{n-1}{2}$. Hence there is an m common to both $\mathcal{A}$ and $\mathcal{B}$. Thus m and $n-m$ are squarefree.

5. Let ε be a positive real number and let $n \in \mathbb{N}$. Show that the number N of different prime factors p of n with $p > n^{\varepsilon}$ satisfies $N \le \frac{1}{\varepsilon}$. Deduce that $\prod_{p|n}\left(1 - \frac{1}{p}\right) \ge c(\varepsilon)\prod_{2\le r\le n^{\varepsilon}}\left(1 - \frac{1}{r}\right)$ where $c(\varepsilon) = 2^{-1/\varepsilon}$. Prove that $c(\varepsilon)n^{1-\varepsilon} \le \phi(n) \le n$. If $N = 0$, then there is nothing to prove. If $N > 0$, then $\log n > 0$ and $n^{\varepsilon N} < \prod_{p|n} p \le n$. Thus $\varepsilon N \log n < \log n$. Now $\prod_{p|n}\left(1 - \frac{1}{p}\right) \ge \left(\prod_{p|n; p>n^{\varepsilon}}\left(1 - \frac{1}{p}\right)\right)\prod_{2\le r\le n^{\varepsilon}}(1 - 1/r)$. The first product on the right is $\ge (1 - 1/2)^N$ and the second telescopes to $1/\lfloor n^{\varepsilon}\rfloor$.

6. Let y be any real number with $y > 1$. By considering the prime divisors p of n with $p > y$, or otherwise, show that $y^{\omega(n)-y} \le n$, i.e. $\omega(n) \le y + \frac{\log n}{\log y}$. Show that $f(x) = 2x^{\frac{1}{2}} - \log x$ is an increasing function of x for $x \ge 1$. Deduce that if $n \ge 3$, then $(\log n)^{\frac{1}{2}} < \frac{2\log n}{\log\log n}$. Prove that if $n \ge 3$, then $\omega(n) \le \frac{4\log n}{\log\log n}$. Crudely the number N of prime divisors p of n with $p \le y$ is at most y. Thus $n \ge \prod_{p|n; p>y} p \ge y^{\omega(n)-y}$. Taking logs gives $\omega(n) \le y + \frac{\log n}{\log y}$. We have $f'(x) = x^{-1/2} - x^{-1} = \frac{x^{1/2}-1}{x} > 0$ when $x > 1$. Thus for $n \ge 3$, since then $\log n > 1$, we have $2(\log n)^{1/2} \ge \log\log n$ and so $(\log n)^{1/2} \le \frac{2\log n}{\log\log n}$. Then taking $y = (\log n)^{1/2}$ gives $\omega(n) \le \frac{4\log n}{\log\log n}$.