567 NUMBER THEORY I, FALL TERM 2008, PROBLEMS 14

Return by Tuesday 9th December

Harder Problems

1. (i) Prove that if $p \equiv 1 \pmod 3$, then $\left(\frac{-3}{p}\right)_L = 1$.
(ii) Let $\mathcal{M} = \{n \in \mathbb{N} : p|n \implies p \equiv 1 \pmod 3\}$. Prove that if $n \in \mathcal{M}$, then $x^2 + 3 \equiv 0 \pmod{4n}$ is soluble in x.
(iii) Let $n \in \mathcal{M}$. Prove that there are a, $B \in \mathbb{Z}$ with $a > 0$ such that $B^2 + 12 = 4an$. Let $b = B - 2a$, $c = (b^2 + 12)/4a$. Prove that $b^2 - 4ac = -12$ and $a + b + c = n$.
(iv) Let $h(d)$ be defined as in homework 11. Prove that $h(-12) = 2$.
(v) Prove that if $n \in \mathcal{M}$, then $x^2 + 3y^2 = n$ is soluble in integers x and y.

2. (i) Prove that if $p \equiv 1, 4 \pmod 7$, then $\left(\frac{-7}{p}\right)_L = 1$.
(ii) Let $\mathcal{N} = \{n \in \mathbb{N} : p|n \implies p \equiv 1, 4 \pmod 7\}$. Prove that if $n \in \mathcal{N}$, then $x^2 + 7 \equiv 0 \pmod{4n}$ is soluble in x.
(iii) Let $n \in \mathcal{N}$. Prove that there are a, $B \in \mathbb{Z}$ with $a > 0$ such that $B^2 + 7 = 4an$. Let $b = B - 2a$, $c = (b^2 + 7)/4a$. Prove that $b^2 - 4ac = -7$ and $a + b + c = n$.
(iv) Recall from homework 11 that $h(-7) = 1$. Prove that if $n \in \mathcal{N}$, then $x^2 + xy + 2y^2 = n$ is soluble in integers x and y.
(v) Let $n \in \mathcal{N}$. Prove that $x^2 + 7y^2 = 4n$ is soluble in integers x, y. Moreover prove that x and y are both even, and thus $x^2 + 7y^2 = n$ is also soluble in intgers x, y.