

**MATH 567 INTRODUCTION TO NUMBER
THEORY I, FALL TERM 2003, PROBLEMS 13**

Return by Thursday 11th December

1. Evaluate (i) $\text{ord}_3(54)$, (ii) $\text{ord}_2(128)$, (iii) $\text{ord}_{13}(-26/169)$, (iv) $\text{ord}_3(9!)$.
2. Evaluate $|a - b|_p$ when (i) $a = 1, b = 26, p = 5$, (ii) $a = 1, b = 244, p = 3$, (iii) $a = 1/9, b = -1/16, p = 5$, (iv) $a = (9!)^2/3^9, b = 0, p = 3$.
3. If $1 \leq a \leq p - 1$, then prove that $\text{ord}_p((ap^N)!) = a(1 + p + p^2 + \cdots + p^{N-1})$.
4. (i) Prove that if $x \in \mathbb{Q}$ and $|x|_p \leq 1$ for every prime $p < \infty$, then $x \in \mathbb{Z}$. (ii) Let $x \in \mathbb{Q} \setminus \{0\}$. Prove that $\prod_p |x|_p = 1$ where the product is over all primes including $p = \infty$.

Let \mathbb{Z}_p be the elements α of \mathbb{Q}_p with $|\alpha|_p \leq 1$. [Thus $\mathbb{Z} \subset \mathbb{Z}_p$.] The form of Hensel's Lemma we will prove in class is as follows. Let $f(x) \in \mathbb{Z}_p[x]$ and suppose that there is an $a \in \mathbb{Z}_p$ such that $|f(a)|_p < |f'(a)|_p^2$, where $f'(x)$ denotes the formal derivative of $f(x)$ (so x^k is replaced by kx^{k-1} for each k). Then there is a $b \in \mathbb{Z}_p$ such that $f(b) = 0$.

5. Show that there is a b in \mathbb{Z}_{13} such that $b^2 = -1$. By observing that there are $r \in \mathbb{Q}$ such that $|r - b|_{13}$ is arbitrarily small deduce that for every $k \in \mathbb{N}$ the congruence $x^2 \equiv -1 \pmod{13^k}$ is soluble.