

567 NUMBER THEORY I, FALL 2008, PROBLEMS 12

Return by Tuesday 18th November

Easier problems

1. Show that for arbitrary real or complex numbers c_1, \dots, c_q ,

$$\sum_{\chi} \left| \sum_{n=1}^q c_n \chi(n) \right|^2 = \varphi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^q |c_n|^2$$

where the sum on the left hand side runs over all Dirichlet characters $\chi \pmod{q}$.

2. Show that for arbitrary real or complex numbers c_{χ} ,

$$\sum_{n=1}^q \left| \sum_{\chi} c_{\chi} \chi(n) \right|^2 = \varphi(q) \sum_{\chi} |c_{\chi}|^2$$

where sums over χ are extended over all Dirichlet characters \pmod{q} .

Harder problem

3. (Mertens (1895a,b)) Let $r(n) = \sum_{d|n} \chi(d)$.

(a) Show that if χ is a non-principal character \pmod{q} , then

$$\sum_{n>x} \frac{\chi(n)}{\sqrt{n}} \ll_{\chi} \frac{1}{\sqrt{x}}.$$

(b) Show that if χ is a non-principal character \pmod{q} , then

$$\sum_{n \leq x} \frac{r(n)}{n^{1/2}} = 2x^{1/2} L(1, \chi) + O_{\chi}(1).$$

(c) Recall that if χ is quadratic then $r(n) \geq 0$ for all n , and that $r(n^2) \geq 1$. Deduce that if χ is a quadratic character, then the left hand side above is $\gg \log x$.

(d) Conclude that if χ is a quadratic character, then $L(1, \chi) > 0$.