MATH 567 NUMBER THEORY I, PROBLEMS 9

To be submitted by Tuesday 28th October

Easier problems

1. $\Omega(n)$ denotes the total number of prime factors on n. Show that $\Omega(n) \leq \frac{\log n}{\log 2}$.

2. Show that $\sum_{n\leq x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x + O(\log x)$ for $x \geq 2$.

3. Let $D(x) = \sum_{n\leq x} d(n)$. Show that $\sum_{n\leq x} \frac{d(n)}{n} = \frac{D(x)}{x} + \int_1^x \frac{D(u)}{u^2}du$. Deduce that $\sum_{n\leq x} \frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + O(\log x)$.

4. A number $n \in \mathbb{N}$ is *squarefree* when it has no repeated prime factors. For $x \in \mathbb{R}$, $x \geq 1$ let $Q(x)$ denote the number of squarefree numbers not exceeding x.

   (i) Show that if $n \in \mathbb{N}$, then $Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor$.

   (ii) Show that $\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}$.

   (iii) Show that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

   (iv) Show that every integer $n > 1$ is a sum of two squarefree numbers.

Harder problems

5. Let ε be a positive real number and let $n \in \mathbb{N}$. Show that the number N of different prime factors p of n with $p > n^{\varepsilon}$ satisfies $N \leq \frac{1}{\varepsilon}$. Deduce that $\prod_{p|n}\left(1 - \frac{1}{p}\right) \geq c(\varepsilon)\prod_{2\leq r\leq n^{\varepsilon}}\left(1 - \frac{1}{r}\right)$ where $c(\varepsilon) = 2^{-1/\varepsilon}$.

   Prove that $c(\varepsilon)n^{1-\varepsilon} \leq \phi(n) \leq n$.

6. Let y be any real number with $y > 1$. By considering the prime divisors p of n with $p > y$, or otherwise, show that $y^{\omega(n)-y} \leq n$, i.e. $\omega(n) \leq y + \frac{\log n}{\log y}$.

   Show that $f(x) = 2x^{\frac{1}{2}} - \log x$ is an increasing function of x for $x \geq 1$. Deduce that if $n \geq 3$, then $(\log n)^{\frac{1}{2}} < \frac{2\log n}{\log\log n}$.

   Prove that if $n \geq 3$, then $\omega(n) \leq \frac{4\log n}{\log\log n}$.