MAT 567 FALL 2008, NUMBER THEORY I, PROBLEMS 8

To be submitted by Tuesday, October 21st

Easier problems

1. Evaluate $\left(\frac{313}{367}\right)_J$, $\left(\frac{367}{401}\right)_J$, $\left(\frac{401}{313}\right)_J$.

2. Show that the congruence $x^6 - 11x^4 + 36x^2 - 36 \equiv 0 \pmod{p}$ is soluble for every prime p. Hint: Factorise $z^3 - 11z^2 + 36z - 36$.

3. Suppose that $a \in \mathbb{Z}\backslash\{0\}$, and there is a $b \in \mathbb{Z}$ such that $a = -b^2$. Show that there is an odd positive integer m such that $\left(\frac{a}{m}\right)_J = -1$. Deduce that there is an odd prime p such that $\left(\frac{a}{p}\right)_J = -1$.

4. Suppose that $a \in \mathbb{Z}\backslash\{0\}$ and $a = \pm 2^u b$ where $u \in \mathbb{N}$ and $b \in \mathbb{N}$ with both u and b odd. Show that there is an odd positive integer m such that $\left(\frac{a}{m}\right)_J = -1$. Deduce that there is an odd prime p such that $\left(\frac{a}{p}\right)_J = -1$. Hint: Let m be a solution to $m \equiv 5 \pmod{8}$, $m \equiv 1 \pmod{b}$.

5. Suppose that $a \in \mathbb{Z}\backslash\{0\}$ and $a = \pm 2^{2u} b q^t$ where u is a non-negative integer, $b \in \mathbb{N}$ and $t \in \mathbb{N}$ with both b and t odd, and q is an odd prime. Show that there is an odd positive integer m such that $\left(\frac{a}{m}\right)_J = -1$. Deduce that there is an odd prime p such that $\left(\frac{a}{p}\right)_J = -1$. Hint: Let m be a solution to $m \equiv 1 \pmod{4b}$, $m \equiv n \pmod{q}$ where n is a quadratic non-residue modulo q.

Harder problem

6. Show that an integer a is a perfect square if and only if it is a quadratic residue for every prime p not dividing a. Questions 3, 4, 5, are relevant. This is a simple example of the “local-to-global” principle.