

MATH 567 FALL 2008 NUMBER THEORY I, PROBLEMS 6

To be submitted by Tuesday, October 7th

Easier problems

1. Show that $\left(\sum_{m|n} d(m)\right)^2 = \sum_{m|n} d(m)^3$.
2. Show that if $\sigma(n)$ is odd, then n is a square or twice a square.
3. Show that $\sum_{l|(m,n)} \mu(l)$ is 1 when $(m,n) = 1$ and is 0 otherwise. Hence prove that $\sum_{m=1; (m,n)=1}^n m = \frac{1}{2}n\phi(n)$ when $n > 1$.
4. Let $\lambda(n) = (-1)^{\Omega(n)}$ (Liouville's function). Show that $\lambda(n) = \sum_{m^2|n} \mu(n/m^2)$.
5. Define $f(n)$ to be $(-1)^{\frac{n-1}{2}}$ when n is odd, 0 when n is even. Show that f is totally multiplicative and is periodic with period 4.

Harder problem

6. Let $k \in \mathbb{N}$, $z \in \mathbb{C}$, $e(\alpha) = \exp(2\pi i\alpha)$. Define $\Phi_k(z) = \prod_{l=1; (l,k)=1}^k (z - e(l/k))$, the k -th cyclotomic polynomial, i.e. the monic polynomial whose roots are the primitive k -th roots of unity.
 - (i) Show that $\prod_{l|k} \Phi_l(z) = z^k - 1$ and $\Phi_1(z) = z - 1$.
 - (ii) Deduce that $\Phi_k(z) = \prod_{l|k} (z^l - 1)^{\mu(k/l)}$.
 - (iii) Show that if $k > 1$, then $\Phi_k(z) = \prod_{l|k} (1 - z^l)^{\mu(k/l)}$ and $\Phi_k(0) = 1$.
 - (iv) By considering the expansion $(1 - z^l)^{-1} = 1 + z^l + z^{2l} + \dots$ when $|z| < 1$ show that $\Phi_k(z)$ has integer coefficients.
 - (v) Let K be the largest squarefree divisor of k . Show that $\Phi_k(z) = \Phi_K(z^{k/K})$.
 - (vi) Prove that $\Phi_p(z) = 1 + z + \dots + z^{p-1}$.
 - (vii) Show that if k is odd and $k > 1$, then $\Phi_{2^r k}(z) = \Phi_k(-z^{2^{r-1}})$.
 - (viii) Suppose that p and q are different primes. Show that, when $|z| < 1$, $\Phi_{pq}(z) = (1-z) \sum_{n=0}^{\infty} b_n z^n$ where b_n is the number of choices of $u, v \in \mathbb{Z}$ with $0 \leq u \leq q-1$, $v \geq 0$ and $up + vq = n$. Deduce that $b_n = 0$ or 1 and that the coefficients of $\Phi_{pq}(z)$ are ± 1 or 0.
 - (ix) Show that if $k < 105$, then the coefficients of $\Phi_k(z)$ are ± 1 or 0.
 - (x) Show that the coefficient of z^7 in Φ_{105} is -2 . It is known (Erdős 1948, Vaughan 1975) that *sometimes* the coefficients of $\Phi_k(z)$ are as large as $\exp\left(2 \frac{\log k}{\log \log k}\right)$ and that "almost always" the largest coefficient is arbitrarily large (Meier 1995). So much for intuition ...!
 - (xi) Prove that if $k > 1$, then $\Phi_k(1) = e^{\Lambda(k)}$.