

**MATH 567 INTRODUCTION TO NUMBER
THEORY I, FALL TERM 2003, PROBLEMS 6**

Return by Thursday 16th October

1. Determine the arithmetic function f such that for every natural number n we have $\mu(n) = \sum_{m|n} f(m)$, i.e. is it multiplicative, and what are its values on the prime powers?
2. Show that every odd number n can be written as the difference of two squares, $n = x^2 - y^2$. How many different choices for the integers x and y are there?
3. Show that if $n > 1$, then

$$\sum_{\substack{m=1 \\ (m,n)=1}}^n m = \frac{1}{2}\phi(n)n$$

4. Show for each positive integer k that there is a unique arithmetic function ϕ_k such that $\sum_{m|n} \phi_k(m) = n^k$. Obtain a formula for $\phi_k(n)$ and show that $\phi_k(n)$ is multiplicative.
5. A natural number n is called *abundant* (*perfect*) if $\sigma(n) \geq 2n$ ($\sigma(n) = 2n$). Prove that any integer of the form $2^{r-1}(2^r - 1)$ with $r \geq 2$ is abundant, and prove also that an *even* number is perfect if, and only if, it is of this form with $2^r - 1$ prime.