MATH 567 FALL 2008, NUMBER THEORY II, PROBLEMS 3

To be submitted by Tuesday, September 16th

Easier problems

1. Show that if p is a prime number and $1 \leq j \leq p-1$, then p divides the binomial coefficient $\binom{p}{j}$.

2. Show that $n|(n-1)!$ for all composite $n > 4$.

3. Exhibit a complete residue system modulo 17 composed entirely of multiples of 3.

4. Solve $11x \equiv 21 \pmod{105}$.

5. Prove that $3n^2 - 1$ can never be a perfect square.

Harder problems

6. Prove that no polynomial $f(x)$ of degree at least 1 with integral coefficients can be prime for every positive integer x.

7. If $2^n + 1$ is an odd prime for some integer n, prove that n is a power of 2.

8. Show that if p is an odd prime, then the number of solutions (i.e., the number of ordered pairs of residues modulo p) of the congruence $x^2 - y^2 \equiv a \pmod{p}$ is $p-1$ when $a \not\equiv 0 \pmod{p}$ and $2p-1$ when $a \equiv 0 \pmod{p}$.