

MATH 567 FALL 2008, NUMBER THEORY I, PROBLEMS 2

To be submitted by Tuesday, September 9th

Easier problems

- Let $a, b, c \in \mathbb{Z}$ with a and b not both zero. Prove each of the following.
 - If $(a, b) = 1$ and $a|bc$, then $a|c$.
 - $\left(\frac{a}{(a,b)}, \frac{b}{(a,b)}\right) = 1$.
 - $(a, b) = (a + cb, b)$.
- Show that if $(a, b) = 1$, then $(a - b, a + b) = 1$ or 2 . Exactly when is the value 2 ?
- Show that if $(b, c) = 1$, then $(a, bc) = (a, b)(a, c)$, and that $(bx + cy, bc) = (b, y)(c, x)$ for all integers x and y .
- Find integers x and y such that $525x + 231y = (525, 231)$.
- Show that if $ad - bc = \pm 1$, then $(a + b, c + d) = 1$.

Harder problem

6. (L. Mirsky and D. J. Newman) Suppose that $K \geq 2$, $0 \leq a_k < m_k$ for $1 \leq k \leq K$ and that $m_1 < m_2 < \dots < m_K$. This is called a *family of covering congruences* when every integer x satisfies at least one of the congruences $x \equiv a_k \pmod{m_k}$. A system of covering congruences is called *exact* when for every value of x there is exactly one value of k such that $x \equiv a_k \pmod{m_k}$. Show that if the system is exact, then

$$\sum_{k=1}^K \frac{z^{a_k}}{1 - z^{m_k}} = \frac{1}{1 - z}.$$

Let $e(\alpha)$ denote $e^{2\pi i \alpha}$ where $i = \sqrt{-1}$. When $z = re(1/m_K)$ with $r \in \mathbb{R}_{>0}$ and $r \rightarrow 1-$, show that the left hand side above is

$$\sim \frac{e(a_K/m_K)}{m_K(1 - r)}$$

whereas the right hand side is bounded for z in a neighbourhood of $e(1/m_K)$.