

**MATH 567 INTRODUCTION TO NUMBER
THEORY I, FALL TERM 2003, PROBLEMS 2**

Return by Tuesday 16th September

1. Find a counter-example to the statement of question 26 on page 16. Correct the question and prove the corrected statement.
2. Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that $m^{1/n}$ is irrational when m is not the n -th power of an integer.
3. Let $\omega = \frac{1}{2}(-1 + \sqrt{-3})$ and define $\mathbb{Z}[\omega] = \{a + b\omega : a \in \mathbb{Z}, b \in \mathbb{Z}\}$. For $\alpha, \beta \in \mathbb{Z}[\omega]$, $\alpha|\beta$ means that there is a $\gamma \in \mathbb{Z}[\omega]$ such that $\alpha\gamma = \beta$. A unit α has the property that $\alpha|\beta$ for every $\beta \in \mathbb{Z}[\omega]$. For $\alpha = a + b\omega \in \mathbb{Z}[\omega]$ define $\lambda(\alpha) = a^2 - ab + b^2$. Show that $\bar{\omega} \in \mathbb{Z}[\omega]$ and that α is a unit of $\mathbb{Z}[\omega]$ iff $\lambda(\alpha) = 1$. Deduce that the units of $\mathbb{Z}[\omega]$ are $1, -1, \omega, -\omega, \omega^2, -\omega^2$ (and that they are distinct).
4. Let $\pi \in \mathbb{Z}[\omega]$. Show that if $\lambda(\pi) = p$ is a prime in \mathbb{Z} , then π is a prime in $\mathbb{Z}[\omega]$.
5. Show that the converse to question 4. is false.