MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, SOLUTIONS 11

1. In class we showed that if $\hat{f} \in L^1(\mathbb{R})$ and f satisfies

$$f(x) = \int_{-b}^{b} \hat{f}(t)e(xt)dt \tag{0}$$

and

$$f(x) = 0 \tag{1}$$

holds for $|x| > a$, then f is identically 0. Prove that the conclusion follows provided only that (1) holds when $x \in (a, a')$ where $a < a'$.

Clearly the integral in (0) is differentiable arbitrarily many times and the k–th derivative satisfies $f^{(k)}(x) = \int_{-b}^{b} \hat{f}(t)e(xt)(2\pi i t)^k dt = 0$ for $x \in (a, a')$. Let $c \in (a, a')$ and $y \in \mathbb{R}$. Then $f(y + c) = \int_{-b}^{b} \hat{f}(t)e(ct)e(yt)dt = \int_{-b}^{b} \hat{f}(t)e(xt) \sum_{k=0}^{\infty} \frac{(2\pi i y t)^k}{k!} dt = \sum_{k=0}^{\infty} \frac{y^k}{k!} \int_{-b}^{b} \hat{f}(t)e(xt)(2\pi i t)^k dt = 0.$

2. An entire function f is of exponential type $T < \infty$ when

$$\limsup_{R\to\infty} R^{-1} \log \max_{|z|=R} |f(z)| = T.$$

(i) Prove that $T < 0$ iff $f \equiv 0$. (ii) Give examples of non-constant entire functions of type 0 and of type 1.

(i) Suppose that $T < 0$. Then we can choose a sequence $\{R_k\}$ of real numbers R_k such that $R_k \to \infty$ as $k \to \infty$ and $\log \max_{|z|=R_k} |f(z)| \le \frac{1}{2} T R_k < 0$. Hence $\max_{|z|=R_k} |f(z)| < 1$ and by the maximum modulus principle f is bounded. But a bounded entire function is a constant (Liouville), say $f(z) = c$. But then $|c| = \max_{|z|=R_k} |f(z)| \le \exp(\frac{1}{2} T R_k) \to 0$ as $k \to \infty$, and so $c = 0$. Now suppose that $f \equiv 0$. Then, for any $R > 0$, $\log \max_{|z|=R} |f(z)| = -\infty$ and so $T = -\infty$. (ii) Any polynomial, not identically 0, is of exponential type 0. Other such functions are expressions like $\sum_{k=0}^{\infty} \frac{z^k}{(2k)!}$. This is bounded by $\exp(\sqrt{|z|})$ and so is also of exponential type 0. The function $f(z) = e^z$ satisfies, for any $R > 0$, $\max_{|z|=R} |f(z)| = \max_{|z|=R} e^{\Re z} = e^R$ and so is of type 1. Multiplying it by any function of order 0 gives another function of order 1.