MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, SOLUTIONS 10

1. This is based on the material in §2.7.1. Suppose that $f \in S \cap L^1(\mathbb{R})$ is given and $u \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfies $u'' - u = -f$.

(i) Show that $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u'(x) = 0$. Hint; the summability of u and f are useful here.

We have $u'' = u - f \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Hence $u'(x) = u'(0) + \int_0^x u''(y)dy \to l_\pm$ as $x \to \pm\infty$. Thus $\|u'\|_\infty < \infty$ and so $u'f \in L^1(\mathbb{R})$. Thus $u''u' - u'u = -u'f$ and $u(x)^2 = u'(x)^2 - u'(0)^2 + u(0)^2 - 2\int_0^x u'(y)f(y)dy$ converges as $x \to \pm\infty$. Hence $|u(x)| \to l'_\pm$ as $x \to \pm\infty$. But $l'_- \neq 0$ or $l'_+ \neq 0$ would imply that $\int_\mathbb{R} |u(x)|dx$ diverges. Hence $l'_\pm = 0$. Moroever $u(y) - u(x) = \int_x^y u'(v)dv$. Letting $y \to +\infty$ gives $u(x) = -\int_x^\infty u'(v)dv$. But if $l_+ \neq 0$ this integral diverges. Hence $l_+ = 0$. Likewise $l_- = 0$.

(ii) Show that $\widehat{(u'')}$ exists and equals $-4\pi^2 t^2 \hat{u}$. Hint; consider integration by parts.

By integration by parts twice $\int_{-R}^R u''(x)e(-xt)dx =$ $\big[u'(x)e(-xt) + 2\pi i t u(x)e(-xt)\big]_{-R}^R - 4\pi^2 t^2 \int_{-R}^R u(x)e(-xt)dx$. Letting $r \to \infty$ aned appealing to part (i) gives the desired conclusion.

(iii) Show that $(1 + 4\pi^2 t^2)\hat{u} = \hat{f}$.

On hypothesis, $\widehat{(u'')} = \hat{u} - \hat{f}$. Substituting this in (ii) and solving for $\hat{f}$ establishes (iii).

(iv) Deduce that $u = \big((1 + 4\pi^2 t^2)^{-1}\hat{f}\big)^\vee = \frac{1}{2}e^{-|x|} \circ f = \frac{1}{2}\int_\mathbb{R} e^{-|x-y|} f(y)dx$. Hint; homework 9, question 2 is useful here.

By (iii) and the continuity of u we have $u = \big((1 + 4\pi^2 t^2)^{-1}\hat{f}\big)^\vee$. Moreover $(1 + 4\pi^2 t^2)^{-1} = \hat{g}(t)$ where $g(x) = \frac{1}{2}e^{-|x|}$. Hence $u = (\hat{g}\hat{f})^\vee$ and $\hat{g}\hat{f} = \widehat{g \circ f}$. Hence $u = g \circ f$ as required.