MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, SOLUTIONS 9

1. (Exercise 2.6.5 [corrected]) Prove that if $f, \hat{f} \in L^1(\mathbb{R})$ and f is continuous on $\mathbb{R}$, then $\lim_{u\to 0+} \big(\exp(-2\pi^2 t^2 u)\hat{f}(t)\big)\check{} = f(x)$ pointwise.

By definition $\big(\exp(-2\pi^2t^2u)\hat{f}(t)\big)\check{} = \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(-2\pi^2t^2u)f(y)e(-yt)e(xt)dydt$ and by Fubini this is $\int_{\mathbb{R}} f(y)\int_{\mathbb{R}}\exp(-2\pi^2t^2u)e(-(y-x)t)dtdy = \int_{\mathbb{R}} f(y)\exp\big(-(x-y)^2/2u\big)dy$. Since $\frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}}\exp\big(-z^2/2u\big)dz = 1$ we have $\big(\exp(-2\pi^2t^2u)\hat{f}(t)\big)\check{} = f(x) + \frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}}\big(f(y)-f(x)\big)\exp\big(-(x-y)^2/2u\big)dy$. Let $\varepsilon > 0$. Choose $\delta > 0$ so that if $|y-x| \le \delta$, then $|f(y)-f(x)| < \varepsilon$. Then $\Big|\frac{1}{\sqrt{2\pi u}}\int_{x-\delta}^{x+\delta}\big(f(y)-f(x)\big)\exp\big(-(x-y)^2/2u\big)dy\Big| \le \frac{1}{\sqrt{2\pi u}}\int_{x-\delta}^{x+\delta}|f(y)-f(x)|\exp\big(-(x-y)^2/2u\big)dy < \varepsilon\frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}}\exp\big(-(x-y)^2/2u\big)dy = \varepsilon$. Moreover $\frac{1}{\sqrt{2\pi u}}\int_{|x-y|\ge\delta}|f(y)|\exp\big(-(x-y)^2/2u\big)dy \le \|f\|_1\frac{1}{\sqrt{2\pi u}}\exp(-\delta^2/2u) \to 0$ as $u \to 0+$ and $\frac{1}{\sqrt{2\pi u}}\int_{|x-y|\ge\delta}|f(x)|\exp\big(-(x-y)^2/2u\big)dy \le 2|f(x)|\frac{1}{\sqrt{2\pi u}}\int_{v\ge\delta}\exp(-v^2/2u)dv \to 0$ as $u \to 0+$. Hence $\limsup_{u\to 0+}\big|\big(\exp(-2\pi^2t^2u)\hat{f}(t)\big)\check{} - f(x)\big| \le \varepsilon$ and this holds for every $\varepsilon > 0$.

2. (Exercise 2.6.6) Prove that if $u > 0$ and $k_u(x) = \frac{u}{\pi(u^2+x^2)}$, then $\hat{k}_u(t) = \exp(-2\pi u|t|)$.

The easiest way to do this is by the Cauchy integral formula. The function $f(z) = \frac{ue(-tz)}{\pi(u^2+z^2)}$ is analytic everywhere in $\mathbb{C}$ except at $z = \pm ui$. When $t > 0$ a contour (semicircle or rectangle) in the lower half–plane gives the integral in question is $-2\pi i$ times the residue at $z = -iu$ and thus is $\exp(-2\pi ut)$. When $t < 0$ a contour in the upper half–plane likewise gives $\exp(2\pi ut)$. To use only the theory of Fourier transforms as given in the course, observe that $\exp(-2\pi u|t|)\check{} = \int_{\mathbb{R}}\exp(-2\pi u|t|)e(xt)dt$ evaluates to $k_u(x)$ and since all the functions are in $L^2(\mathbb{R})$ and are continuous, it follows that $\hat{k}_u(t) = \exp(-2\pi u|t|)$.

3. (Exercise 2.6.7) Use the previous exercise to prove that if $f \in L^1(\mathbb{R})$, then $\lim_{u\to 0+}\|(\exp(-2\pi u|t|)\hat{f}(t))\check{} - f(x)\|_1 = 0$.

We have $(\exp(-2\pi u|t|)\hat{f}(t))\check{}(x) = \int_{\mathbb{R}}\exp(-2\pi u|t|)e(xt)\big(\int_{\mathbb{R}}f(y)e(-yt)dy\big)dt$. By Fubini this is $\int_{\mathbb{R}}f(y)\left(\exp(-2\pi u|t|)e(t(x-y))dt\right)dy = \int_{\mathbb{R}}f(y)\frac{u}{\pi(u^2+(x-y)^2)}dy$. Hence $(\exp(-2\pi u|t|)\hat{f}(t))\check{}(x) - f(x) = \int_{\mathbb{R}}(f(x+v)-f(x))\frac{u}{\pi(u^2+v^2)}dv$ and so $\|(\exp(-2\pi u|t|)\hat{f}(t))\check{}(x) - f(x)\|_1 \le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|f(x+v)-f(x)|\frac{u}{\pi(u^2+v^2)}dv\Big)dx$ and by Fubini this is $\int_{\mathbb{R}}\big(\int_{\mathbb{R}}|f(x+v)-f(x)|dx\big)\frac{u}{\pi(u^2+v^2)}dv$. Since the translation operator is continuous on $L^1(\mathbb{R})$, given $\varepsilon > 0$ we can find a $\delta > 0$ such that $\int_{\mathbb{R}}|f(x+v)-f(x)|dx < \varepsilon$ whenever $|v| \le \delta$. Hence $\int_{|v|\le\delta}\big(\int_{\mathbb{R}}|f(x+v)-f(x)|dx\big)\frac{u}{\pi(u^2+v^2)}dv \le \varepsilon\int_{\mathbb{R}}\frac{u}{\pi(u^2+v^2)}dv = \varepsilon$. Moroever $\int_{|v|\ge\delta}\big(\int_{\mathbb{R}}|f(x+v)-f(x)|dx\big)\frac{u}{\pi(u^2+v^2)}dv \le 2\|f\|_1\int_{|v|\ge\delta}\frac{u}{\pi(u^2+v^2)}dv \le 4\|f\|_1 u/\delta \to 0$ as $u \to 0+$. Hence $\limsup_{u\to 0+}\|(\exp(-2\pi u|t|)\hat{f}(t))\check{}(x) - f(x)\|_1 \le \varepsilon$ and this is true for every $\varepsilon > 0$.