MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, SOLUTIONS 8

1. Let $L^1(\mathbb{R}^+, x^{-1}dx)$ denote the set of functions f defined on the positive real numbers $\mathbb{R}^+$ such that $\int_0^\infty |f(x)|x^{-1}dx < \infty$. (i) Show that the convolution product $(f \circ g)(x) = \int_0^\infty f(x/y)g(y)y^{-1}dy$ induces an algebra on $L^1(\mathbb{R}^+, x^{-1}dx)$. (ii) Define the "Mellin transform" formally by $f^\sharp(t) = \int_0^\infty f(x)x^{-2\pi it}x^{-1}dx$. Prove that the map $T : f(\cdot) \to f(e^\cdot)$ establishes a $1:1$, linear, product–preserving transformation from $L^1(\mathbb{R}^+, x^{-1}dx)$ onto the convolution algebra on $L^1(\mathbb{R})$ and that $f^\sharp = \widehat{(Tf)}$.

(i) That $\circ$ is commutative follows immediately by the change of variable $y = x/z$. Moreover for $f, g, h \in L^1(\mathbb{R}^+, x^{-1}dx)$, $(f\circ g)\circ h = \int_0^\infty (f\circ g)(z)h(z)z^{-1}dz = \int_0^\infty (g\circ f)(z)h(z)z^{-1}dz = \int_0^\infty \left(\int_0^\infty g(x/yz)f(y)y^{-1}dy\right) h(z)z^{-1}dz$, and by Fubini this is $\int_0^\infty \left(\int_0^\infty g(x/yz)h(z)z^{-1}dz\right) f(y)y^{-1}dy = (g \circ h) \circ f = f \circ (g \circ h)$. (ii) The change of variable $x = e^y$ gives a convolution product preserving map from $L^1(\mathbb{R}^+, x^{-1}dx)$ to $L^1(\mathbb{R})$ and the change of variable $x = \log y$ gives an inverse map. Let $g(x) = Tf(x) = f(e^x)$. Then $f^\sharp(t) = \int_0^\infty f(x)x^{-1-2\pi it}dx = \int_{-\infty}^\infty f(e^y)e(-yt)dy = \widehat{Tf}(t)$. On the other hand, given $g \in L^1(\mathbb{R})$ define $f(y) = g(\log y)$. Then $\widehat{Tf}(t) = \hat{g}(t) = \int_{-\infty}^\infty g(x)e(-xt)dx = \int_0^\infty f(y)y^{-1-2\pi it}dy = f^\sharp(t)$.

2. Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$. (i) Prove that $\int_a^b \hat{f}(t)e(xt)dt = (f \circ D)(x)$ where $D(u) = \frac{e(bu)-e(au)}{2\pi iu}$. (ii) Prove that if $\int_{-1}^1 |f(x+y) - f(x)|\frac{dy}{|y|} < \infty$, then $f(x) = \lim_{a\to-\infty} \int_a^0 \hat{f}(t)e(xt)dt + \lim_{b\to\infty} \int_0^b \hat{f}(t)e(xt)dt$.

(i) We have $\int_a^b \hat{f}(t)e(xt)dt = \int_a^b \left(\int_\mathbb{R} f(y)e(-yt)dy\right) e(xt)dt$. By Fubini this is $\int_\mathbb{R} f(y)\left(\int_a^b e(t(x-y))dt\right) dy = \int_\mathbb{R} f(y)\frac{e(b(x-y))-e(a(x-y))}{2\pi(x-y)}dy = \int_\mathbb{R} f(y)D(x-y)dy = \int_\mathbb{R} f(x-z)D(z)dz$. When $a < 0 < b$ we have $\int_\mathbb{R} D(z)dz = 1$. Hence $\int_a^b \hat{f}(t)e(xt)dt - f(x) = \int_\mathbb{R}(f(x-z)-f(x))D(z)dz$. By integration by parts, $\int_{|x|\ge1} D(z)dz \ll \frac{1}{|b|}+\frac{1}{|a|}$ and since $f \in L^1(\mathbb{R})$ we have $\int_{|z|\ge1} f(x-z)z^{-1}e(wz)dz \to 0$ as $|w| \to \infty$ by the Riemann–Lebesgue lemma. Hence $\int_{|z|\ge1}(f(x-z) - f(x))D(z)dz \to 0$ as $b \to \infty$ and $a \to -\infty$. Finally $\int_{|z|\le1}(f(x-z) - f(x))D(z)dz = \int_{|z|\le1} \frac{f(x-z)-f(x)}{2\pi iz}(e(bz) - e(az))dz$ and again by the Riemann–Lebesgue lemma and the hypothesis this $\to 0$ as $b \to \infty$ and $a \to -\infty$.