Math 504 Analysis in Euclidean Spaces, Spring Term 2009, Solutions 7

1. Prove that if f, $g \in L^1(\mathbb{R})$, then $\|f \circ g\|_1 \le \|f\|_1 \|g\|_1$.

By definition $(f \circ g)(x) = \int_{\mathbb{R}} f(y) g(x-y) dy$. Hence by the triangle inequality, $|(f \circ g)(x)| \le \int_{\mathbb{R}} |f(y) g(x-y)| dy$ and $\|f \circ g\|_1 = \int_{\mathbb{R}} |(f \circ g)(x)| dx \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y) g(x-y)| dy \right) dx$. By Fubini the RHS is $\int_{\mathbb{R}} |f(y)| \left(\int_{\mathbb{R}} |g(x-y)| dx \right) dy$. The inner integral is $\|g\|_1$ by an obvious change of variable and thus the whole is $\|f\|_1 \|g\|_1$ as required. The fact that this is an upper bound for everything justifies the manipulations.

2. Let $X > 0$ and define $f(x) = \max\left(0, 1 - \frac{|x|}{X}\right)$. Prove that

$$\hat{f}(t) = \begin{cases} \frac{1}{X} \left(\frac{\sin \pi X t}{\pi t} \right)^2 & (t \ne 0), \\ X & (t = 0) \end{cases}$$

and that the inverse Fourier transform of $\hat{f}$ is f. First a slight simplification – not necessary but it slightly simplifies some formulae. Put $f_X(x) = f(x)$. Then a change of variable, $y = Xx$ gives $\hat{f}_X(t) = \hat{f}_1(t/X)$, so we can assume $X = 1$ henceforward. Extend the function f to $\mathbb{C}\backslash\{0\}$ by taking $f(z) = (\pi z)^{-2} \sin^2 \pi z$. Then f has a removeable singularity at $z = 0$ and $\lim_{z \to 0} = 1$. Thus f is essentially entire. Since $f(-x) = f(x)$ it suffices to evaluate the integral when $t \ge 0$. Let $L(R, \varepsilon)$, with R large and ε small, denote the path consisting of the line segments from $-R$ to $-\varepsilon$ and from ε to R, and the semicircle of radius ε from $-\varepsilon$ to ε *via* $-i\varepsilon$. Then $\lim_{\varepsilon \to 0} \lim_{R \to \infty} \int_{L(R,\varepsilon} f(z) e(-zt) dz = \hat{f}(t)$. Write the integrand as $\left(2e^{-2\pi i z t} - e^{2\pi i z(1-t)} - e^{2\pi i z(-1-t)}\right) \left(4\pi^2 z^2\right)^{-1} = g_1(z) - g_2(z) - g_3(z)$ where $g_1(z) = \frac{2e^{-2\pi i z t}}{4\pi^2 z^2}$, $g_2(z) = \frac{e^{2\pi i z(1-t)}}{4\pi^2 z^2}$, $g_3(z) = \frac{e^{2\pi i z(-1-t)}}{4\pi^2 z^2}$. Let C_R^- denote the semicircle of radius R from R to $-R$ *via* $-iR$ and let C_R^+ denote the semicircle of radius R from R to $-R$ *via* iR. Then the g_j are analytic on and inside the contour $L(R, \varepsilon) + C_R^-$. Moreover when $t \ge 0$, $\int_{C_R^-} g_1(z) dz$ and $\int_{C_R^-} g_3(z) dz$ both $\to 0$ as $R \to \infty$, and likewise for $\int_{C_R^-} g_2(z) dz$ when $t \ge 1$. Thus in each case, by Cauchy's theorem, $\int_{L(R,\varepsilon)} g_j(z) dz \to 0$ as $R \to \infty$. When $0 \le t < 1$, $g_2(z)$ is analytic in $\mathbb{C}$ except at $z = 0$ where it has a double pole with residue $\frac{t-1}{2\pi i}$. Moreover $\int_{C_R^+} g_2(z) dz \to 0$ as $R \to \infty$. Thus, by the residue theorem, $\int_{L(R,\varepsilon)} g_2(z) dz \to t - 1$ as $R \to \infty$.

The inverse transform is $\int_{-X}^0 (1 + t/X) e(xt) dt + \int_0^X (1 - t/X) e(xt)$ and these integrals are easily computed by integrating by parts.

3. Prove that (i) $\exp(-x^2)$ belongs to the Schwartz class but that (ii) $\frac{1}{1+x^2}$ and (iii) $\exp(-|x|)$ do not.

(i) It is easily proved by induction on k that $\frac{d^k}{dx^k} \exp(-x^2) = P_k(x) \exp(-x^2)$ where $P_k(x)$ is a polynomial of degree k. Hence, for any pair of non–negative integers j, k, $x^j \frac{d^k}{dx^k} \exp(-x^2) \to 0$ as $|x| \to \infty$. (ii) $e^{-|x|}$ is not differentiable at $x = 0$. (iii) $x^2 (1 + x^2)^{-1} \not\to 0$ as $|x| \to \infty$ (for example).