Math 504 Analysis in Euclidean Spaces, Spring Term 2009, Solutions 6

1. Suppose that $\{u_n\}$ is uniformly distributed (mod 1), and let c be a real number. Put $v_n = u_n + c$. Show that $\{v_n\}$ is uniformly distributed.

By Weyl I, for $h \in \mathbb{N}$ we have $\frac{1}{n}\sum_{m=1}^{n} e(hu_m) \to 0$ as $n \to \infty$. Hence $\frac{1}{n}\sum_{m=1}^{n} e(hv_n) = e(hc)\frac{1}{n}\sum_{m=1}^{n} e(hu_m) \to 0$ as $n \to \infty$.

2. Let $\alpha_n = \log n - \lfloor \log n \rfloor$. (a) Show that $\limsup_{N\to\infty} \frac{1}{N}\text{card}\{n : 1 \le n \le N, \alpha_n \in [0,1/2]\} = \frac{e-e^{1/2}}{e-1}$. (b) Show that $\liminf_{N\to\infty} \frac{1}{N}\text{card}\{n : 1 \le n \le N, \alpha_n \in [0,1/2]\} = \frac{e^{1/2}-1}{e-1}$. (c) Show that $\frac{1}{N}\sum_{n=1}^{N} e(k \log n) = \frac{N^{2\pi i k}}{2\pi i k+1} + O\Big(|k|\frac{1+\log N}{N}\Big)$. (d) Show that the sequence $\{\alpha_n\}$ is not uniformly distributed (mod 1).

(a), (b). Let $L \in \mathbb{N}$ and $N \in \mathbb{N}$ be such that $\lfloor \log N \rfloor = L$. We sort the $n \le N$ according to the value of $k = \lfloor \log n \rfloor$. Then $0 \le k \le \log N$ and $k \le \log n < k+1$. If $\alpha_n \in [0,1/2]$, then $k \le \log n \le k + \frac{1}{2}$ and so we are counting the n with $e^k \le n \le \min(N, e^{k+1/2})$. This is $\lfloor e^{k+1/2} \rfloor - \lceil e^k \rceil + 1$ when $0 \le k \le L-1$ and $\min(N, \lfloor e^{L+1/2} \rfloor) - \lceil e^L \rceil + 1$ when $k = L$. This is $e^k(e^{1/2}-1) + O(1)$ and $\min(N, e^{L+1/2}) - e^L + O(1)$ respectively. Thus $\text{card}\{n : 1 \le n \le N, \alpha_n \in [0,1/2]\} = \min(N, e^{L+1/2}) - e^L + \sum_{k=0}^{L-1} e^k(e^{1/2}-1) + O(\log N) = \min(N, e^{L+1/2}) - e^L\frac{e-e^{1/2}}{e-1} + O(\log N)$. We have $N = e^{L+\theta}$ for some θ with $0 \le \theta < 1$. Then $\frac{1}{N}\text{card}\{n : 1 \le n \le N, \alpha_n \in [0,1/2]\} = 1 - \frac{e-e^{1/2}}{e^{\theta}(e-1)} + O((\log N)/N)$ when $\theta \le 1/2$ and $= \frac{e^{3/2}-e}{e^{\theta}(e-1)} + O((\log N)/N)$ when $\frac{1}{2} < \theta$. The main term here lies between $\frac{e^{1/2}-1}{e-1}$ and $\frac{e-e^{1/2}}{e-1}$ with equality when $\theta = 0$ and $\frac{1}{2}$ respectively. Moreover it is a continuous function of θ. By taking $N = \lceil e^L \rceil$ and $N = \lfloor e^{L+1/2} \rfloor$ respectively we obtain θ_L tending to 0 and 1/2 respectively and thus the upper and lower limits for $\frac{1}{N}\text{card}\{n : 1 \le n \le N, \alpha_n \in [0,1/2]\}$ are achieved in the limits.

(c) The general term here is $n^{2\pi i k} = N^{2\pi i k} - \int_n^N 2\pi i k u^{2\pi i k - 1} du$ and so the sum in question is $N^{1+2\pi i k} - \int_1^N 2\pi i k \lfloor u \rfloor u^{2\pi i k-1} du$. Here the integral is $\int_1^N 2\pi i k u^{2\pi i k} du + O(|k| \log N)$ and integrating the integral gives the desired conclusion.

(d) Any of (a), (b), (c) shows that α_n is not uniformly distributed modulo one. It is only necessary to observe that (a) $\frac{e-e^{1/2}}{e-1} \ne \frac{1}{2}$, (b) $\frac{e^{1/2}-1}{e-1} \ne \frac{1}{2}$, or (c) $N^{2\pi i k} \not\to 0$ as $N \to \infty$.

3. Suppose that the sequence α_n satisfies $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n) = \beta$. Prove that if $\beta \in \mathbb{R}\backslash\mathbb{Q}$ then α_n is uniformly distributed modulo 1.

We have $\alpha_{n+1} = \alpha_n + \beta + \delta_n$ where $\delta_n \to 0$ as $n \to \infty$. Fix $h \in \mathbb{N}$. Then $\sum_{m=1}^{n} e(h\alpha_{n+1}) = \sum_{m=1}^{n} e(h(\alpha_m + \beta + \delta_m))$ and $e(h\delta_m) = 1 + O(\delta_m)$. Hence $\sum_{m=1}^{n} e(h(\alpha_m + \beta + \delta_m)) = e(h\beta)\sum_{m=1}^{n} e(h\alpha_m) + O(\sum_{m=1}^{n} |\delta_m|)$. Also, $\sum_{m=1}^{n} e(h\alpha_{m+1}) = \sum_{m=1}^{n} e(h\alpha_m) + O(1)$. Since $\beta \in \mathbb{R}\backslash\mathbb{Q}$ we have $e(h\beta) \ne 1$. Thus $\big(1 - e(h\beta)\big)\sum_{m=1}^{n} e(h\alpha_m) \ll \frac{1}{n} + \frac{1}{n}\sum_{m=1}^{n} |\delta_m| \to 0$ as $n \to \infty$.